

FLUCTUATIONS OF RECTANGULAR YOUNG DIAGRAMS OF INTERLACING WIGNER EIGENVALUES

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ABSTRACT. We prove a new CLT for the *difference* of linear eigenvalue statistics of a Wigner random matrix H and its minor \hat{H} and find that the fluctuation is much smaller than the fluctuations of the individual linear statistics, as a consequence of the strong correlation between the eigenvalues of H and \hat{H} . In particular our theorem identifies the fluctuation of Kerov's rectangular Young diagrams, defined by the interlacing eigenvalues of H and \hat{H} , around their asymptotic shape, the Vershik-Kerov-Logan-Shepp curve. This result demonstrates yet another aspect of the close connection between random matrix theory and Young diagrams equipped with the Plancherel measure known from representation theory. For the latter a CLT has been obtained in [20] which is structurally similar to our result but the variance is different, indicating that the analogy between the two models has its limitations. Moreover, our theorem shows that Borodin's result [7] on the convergence of the spectral distribution of Wigner matrices to a Gaussian free field also holds in derivative sense.

1. INTRODUCTION

There is a rich history of probabilistic models of essentially algebraic nature with surprising connections to random matrix theory. Examples include the longest increasing subsequence in random permutations [3], queuing processes [6], random tilings of a hexagon [24], poly-nuclear growth processes [36] and 1+1 dimensional exclusion processes (see e.g. [9] for a good overview of the topic). Recent years have seen a spectacular progress towards the KPZ universality that is detected in the extreme regimes. The intuition for the KPZ universality often comes from relating these model to the extreme eigenvalues of random matrices. Unlike in random matrices, the bulk regime in these models received somewhat less attention. Nevertheless, it is an interesting question to examine to what extent the connection to random matrices extends to the bulk regime. Many of these models are related to a classical algebraic problem, the statistics of Young tableaux from representation theory. In this paper we investigate the analogy between large Young diagrams equipped with the classical Plancherel measure and Kerov's *rectangular Young diagrams*, originating from eigenvalue statistics of minors of large random Wigner random matrices. Their limiting shape curves coincide. Here we identify the fluctuation of the rectangular Young diagrams. We find that it is Gaussian on the same scale as the already known fluctuations of the classical Young diagrams, but with a different variance. In particular, the limiting behavior of the two diagram ensembles are not the same, even though in the extreme regime their statistics coincide.

Given an integer N and a partition $N = \lambda_1 + \lambda_2 + \dots$ of N into integers $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, the planar figure obtained as the union of consecutive rows consisting of $\lambda_1, \lambda_2, \dots$ unit square cells, is called the *Young diagram* corresponding to λ of size $|\lambda| = N$. Young diagrams of size N are commonly considered as a probability space equipped with the *Plancherel measure* $P_N(\lambda) := d_\lambda^2/N!$, where d_λ is the number of Young tableaux with given shape λ (see, e.g. [16]).

The first major connection between Young diagrams and random matrix theory was established by Baik, Deift and Johansson who showed in [3] that the distribution of λ_1/\sqrt{N} with respect to $P_N(\lambda)$ asymptotically agrees with the distribution of the largest eigenvalue of an $N \times N$ GUE matrix, hence it follows the Tracy Widom law [40]. Similar result [4] holds for λ_2/\sqrt{N} and the second largest eigenvalue, and Okounkov [33] established that the joint distribution of $\lambda_1/\sqrt{N}, \dots, \lambda_k/\sqrt{N}$ asymptotically follows that of the k largest eigenvalues of the GUE. Alternative proofs are given in [11, 23].

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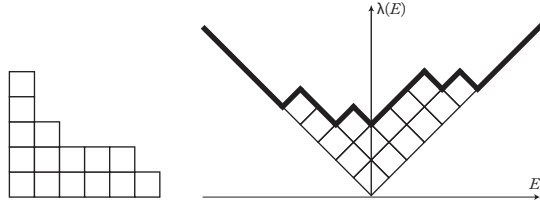


FIGURE 1. Young diagram in French and Russian convention corresponding to the partition $15 = 6 + 5 + 2 + 1 + 1$, together with the curve $\lambda(E)$

To study the bulk behavior of Young diagrams, it is convenient to draw them in the Russian convention which is rotated by 45° from the horizontal convention (see Figure 1). In this way we can view the upper boundary the diagram as a continuous function $E \mapsto \lambda(E)$ such that $\lambda(E) \geq |E|$ and $\lambda'(E) = \pm 1$, whenever it is defined. We can continuously extend this function by $\lambda(E) = |E|$ outside the extent of the diagram. The limiting shape and the fluctuation of this curve under the Plancherel measure, after proper rescaling, has been determined:

$$\frac{1}{\sqrt{N}}\lambda(\sqrt{N}E) \approx \Omega(E) + \frac{2}{\sqrt{N}}\Delta(E), \quad N \rightarrow \infty, \quad (1)$$

where

$$\Omega(E) := \begin{cases} |E| & \text{if } |E| \geq 2 \\ \frac{2}{\pi} \left[E \arcsin \frac{E}{2} + \sqrt{4 - E^2} \right] & \text{else} \end{cases}$$

is the *Vershik-Kerov-Logan-Shepp curve*. The fluctuation term $\Delta(E)$ is a generalized Gaussian process on the interval $[-2, 2]$ that can be defined by the trigonometric series

$$\Delta(2 \cos \theta) = \frac{1}{\pi} \sum_{k \geq 2} \frac{\xi_k \sin k\theta}{\sqrt{k}}$$

of independent standard Gaussian random variables ξ_k . The limit shape has been independently identified in [31] and [27], the fluctuation was proved in [20] following Kerov's unpublished notes.

A direct connection between random matrices and Young diagrams in the bulk regime was found by Kerov in [28]. He showed that for a Wigner random matrix $H \in \mathbb{C}^{N \times N}$ and an independent random $N - 1$ dimensional hyperplane h with uniformly distributed normal vector, the eigenvalues $\lambda_1, \dots, \lambda_N$, and $\hat{\lambda}_1, \dots, \hat{\lambda}_{N-1}$ of H and $P_h H P_h$, where P_h is the projection onto h , can be used to construct a curve very similar to Young diagrams. He defined a *rectangular Young diagram* (for a more general context, see [34]) as the function

$$w_N(E) := \sum_{k=1}^N |\lambda_k - E| - \sum_{k=1}^{N-1} |\hat{\lambda}_k - E|, \quad E \in \mathbb{R}.$$

It is easy to see that w_N is the unique piecewise-linear continuous function with local minima in $\lambda_1 \leq \dots \leq \lambda_N$ and local maxima in $\hat{\lambda}_1 \leq \dots \leq \hat{\lambda}_{N-1}$ such that the slope is ± 1 whenever it exists and $w_N(E) = \left| E - \sum \lambda_k + \sum \hat{\lambda}_k \right|$ for large enough $|E|$. It was then shown that

$$\lim_{N \rightarrow \infty} \mathbf{E} w_N(E) = \Omega(E),$$

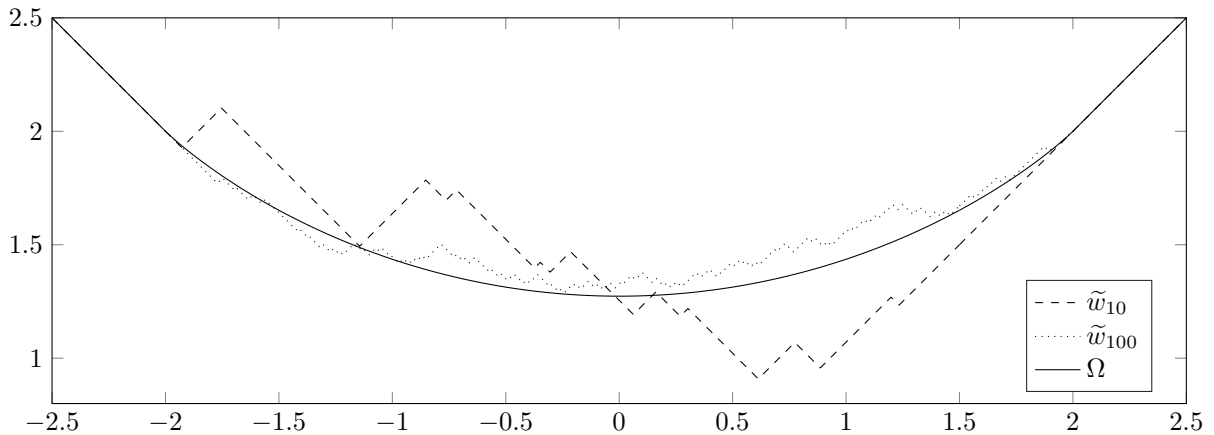
uniformly in E .

Bufetov in [12] has recently improved this result in two directions. First, he showed that the randomness in the choice of the projection is not needed; it is sufficient to consider the eigenvalues of H and its minor $\hat{H} = (h_{ij})_{i,j \geq 2}$ (where the choice of removed row/column is, of course, arbitrary). Second, he improved the convergence in expectation to convergence in probability;

$$\lim_{N \rightarrow \infty} \sup_E |w_N(E) - \Omega(E)| = 0. \quad (2)$$

We note that $\sum_k \lambda_k = \sum_k \hat{\lambda}_{k-1} + h_{11}$, so $w_N(E) = |E - h_{11}|$ for large E and thus it does not exactly match $\Omega(E)$ even outside of the limiting spectrum $[-2, 2]$. To remedy this, we will also consider the *shifted diagram*

$$\tilde{w}_N(E) := w_N(E + h_{11})$$

FIGURE 2. Sample rectangular Young diagrams \tilde{w}_N with limiting shape Ω

which agrees identically with $\Omega(E)$ outside the spectrum. This modification is irrelevant for the limit shape but it becomes relevant when we consider fluctuations. Figure 2 shows realizations of \tilde{w}_N for different values of N together with the limiting curve Ω .

In the present work we improve the law of large numbers type results (2) to a central limit theorem (CLT) as in (1), and thus demonstrate that the analogy between random matrices and representation theory extends beyond the macroscopic behavior. Specifically, we prove that

$$w_N(E) \approx \Omega(E) + \frac{1}{\sqrt{N}}F(E) \quad \text{and} \quad \tilde{w}_N(E) \approx \Omega(E) + \frac{1}{\sqrt{N}}\tilde{F}(E) \quad (3)$$

where $F(E), \tilde{F}(E)$ are collections of centered random variables and we compute their covariance structure. Moreover, we show that the distribution of $F(E)$ and $\tilde{F}(E)$ is Gaussian if and only if h_{11} is Gaussian. We conclude from our explicit formulas for the variances that although (3) resembles (1) closely, the distributions of $F(E), \tilde{F}(E)$ and $\Delta(E)$ do not agree even if h_{11} is Gaussian. For example – in contrast to $\Delta(E)$ – the fluctuation terms $F(E), \tilde{F}(E)$ have a finite variance.

We prove these results as corollaries to a new central limit theorem for the *difference* in linear eigenvalue statistics of a Wigner random matrix and its minor. For many classes of random matrices $H = H^{(N)} \in \mathbb{C}^{N \times N}$ the empirical spectral density, i.e., the normalized counting measure of eigenvalues, $\frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k}$ converges weakly to a deterministic measure ρ as $N \rightarrow \infty$, which may be viewed as type of law of large numbers. Phrased in terms of an appropriate test function f ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(\lambda_k(H^{(N)})) = \int f(x) \rho(dx),$$

naturally raises the question whether the fluctuations in this convergence also follow an analogue of the central limit theorem. The object $\sum_{k=1}^N f(\lambda_k(H^{(N)})) = \text{Tr } f(H^{(N)})$, called the *linear eigenvalue statistics* of $H^{(N)}$, has been studied for many types of random matrices [32, 2, 38, 37, 21, 5, 39] and large classes of test functions f . Contrary to the classical CLT, the fluctuations of the linear eigenvalue statistics do not grow with N , at least if f is sufficiently regular. The fluctuations are typically Gaussian, but there are also some pathological examples where this is not the case, e.g. for certain invariant ensembles with density supported on several intervals [35]. For polynomial test function f the Gaussian fluctuation can be proved by the elementary moment method, see e.g. [1, Theorem 2.1.13], but a simple approximation argument does not suffice to extend the result to less regular f . CLT still holds, for example it has been shown in [32] that for GOE random matrices and test functions f with bounded derivative $\text{Tr } f(H^{(N)}) - \mathbf{E} \text{Tr } f(H^{(N)})$ converges in distribution to a centered Gaussian random variable of variance

$$\frac{1}{2\pi^2} \int_{-2}^2 \int_{-2}^2 \left(\frac{f(x) - f(x')}{x - x'} \right)^2 \frac{4 - xx'}{\sqrt{4 - x^2} \sqrt{4 - x'^2}} dx dx'. \quad (4)$$

The currently weakest regularity conditions on f for CLT are found in [39]; $f \in H^{1+\epsilon}$ is necessary for general Wigner matrices and $f \in C^{1/2+\epsilon}$ suffices for GUE. We stress that linear statistics are very sensitive to regularity

of the test function; while polynomial test functions do not require understanding of any local eigenvalue statistics (the global moment method works), the proof in [39] for the Wigner case heavily relied on techniques developed to prove local semicircle laws [15], while the GUE case even used the Brézin-Hikami formula and saddle point analysis of the determinantal kernel by Johansson [22].

All previous work concerned linear statistics of a single Wigner matrix except two papers by Borodin [7, 8] and a few recent works motivated by them. In these papers joint fluctuations of linear statistics of Wigner matrices and its minors were investigated (see also [25] where a similar question was discussed for d -regular graphs). Borodin considered general families of regularly nested minors and identified the limit of their joint spectral counting functions as a Gaussian free field (GFF), but the test function was polynomial and thus a relatively simple extension of the moment method [1] worked. The class of test functions was extended to include functions with a high Sobolev regularity ($H^{2.5+\epsilon}$ for Gaussian and $H^{5.5+\epsilon}$ for general Wigner matrices) using a Chebyshev basis decomposition [30] (see also [26] where not only nested but overlapping matrices were considered). However, all these results identify the joint fluctuations on order one scale, whose correlations are typically strictly between 0 and 1 for a collection of minors whose sizes asymptotically differ by cN . Our work detects the small fluctuation of order $N^{-1/2}$ resulting from the very strong correlation between minors of almost the same size. This fine effect is not visible on the scale of the analysis in [7, 8, 30]. Nevertheless, one may ask whether the fine scale covariance structure proven in our main Theorem 1 is consistent with the covariance formula in [7, 8, 30] if one formally applies it to H and its immediate minor \widehat{H} disregarding the interchange of limits. Effectively this question is equivalent to asking whether the convergence of the spectral counting functions of the minors to the GFF also holds in derivative sense. In Appendix A we show that the derivative of the GFF predicts the correct variance of the fluctuations but fails to identify their distribution, in general. This is essentially due to the fact that our fine scale result depends on the precise distribution of h_{11} while the macroscopic formula does not depend on any individual matrix entry.

Motivated by Kerov's rectangular Young diagrams, in the present work we study the difference of two linear statistics $f_N := \text{Tr } f(H) - \text{Tr } f(\widehat{H})$ of a Wigner matrix and its minor for a large class of test functions that includes $f(x) = |x - E|$. We find that the expectation of f_N converges to

$$\Omega_f := \frac{1}{\pi} \int_{-2}^2 \frac{f(x)}{\sqrt{4-x^2}} dx$$

and its fluctuations around Ω_f are of order $N^{-1/2}$. In particular, the fluctuations we identify are much smaller than those of the individual linear statistics, as a result of the strong correlation of the eigenvalues of H and \widehat{H} . Moreover, we prove that the fluctuations are Gaussian if and only if h_{11} follows a normal distribution. It is clear that h_{11} plays a special role, since for example with $f(x) = x$, we have $f_N = \text{Tr } H - \text{Tr } \widehat{H} = h_{11}$. Since our test function has a relatively low regularity, our proof requires to understand the spectral statistics on small mesoscopic scales. In practice, we jointly analyze the Green functions $G(z) = (H - z)^{-1}$ and $\widehat{G}(z) = (\widehat{H} - z)^{-1}$ on a spectral scale $\Im z \geq N^{-2/3}$.

After completing this manuscript, we learned¹ from Vadim Gorin that he and Lingfu Zhang have obtained [17] the exact analogue of our result for the multilevel extension of the β -Jacobi ensemble that was introduced in [10] as an analogue of the minor process for general β -ensemble.

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2. MAIN RESULTS

We consider complex Hermitian and real symmetric random matrices and their minors of the form

$$H := \begin{pmatrix} h_{11} & \cdots & h_{N1} \\ \vdots & \ddots & \vdots \\ h_{1N} & \cdots & h_{NN} \end{pmatrix}, \quad \widehat{H} := \begin{pmatrix} h_{22} & \cdots & h_{N2} \\ \vdots & \ddots & \vdots \\ h_{2N} & \cdots & h_{NN} \end{pmatrix}$$

with $(h_{ij})_{i,j=1}^N$ being independent (up to the symmetry constraint $h_{ij} = \overline{h_{ji}}$) random variables satisfying

$$\mathbf{E} h_{ij} = 0, \quad \mathbf{E} |h_{ij}|^2 = \frac{s_{ij}}{N} \quad \text{and} \quad \mathbf{E} |h_{ij}|^p \leq \frac{\mu_p}{N^{p/2}} \quad (5)$$

for all i, j, p and some absolute constants μ_p . Our main result about the difference of linear eigenvalue statistics of a Wigner random matrix and its minor is as follows.

¹Private communication

Theorem 1. *Let the Wigner matrix H satisfy (5), $s_{ij} = 1$ for $i \neq j$ and $s_{ii} \leq C$ for all i , $\mathbf{E}|h_{1j}|^4 = \sigma_4/N^2$ for $j = 2, \dots, N$ and $\mathbf{E}h_{ij}^2 = \sigma_2/N$ for $i < j$. Moreover, let $f \in H^2([-10, 10])$ be some real-valued function. Then the random variables*

$$f_N := \text{Tr } f(H) - \text{Tr } f(\widehat{H}) = \sum_{k=1}^N f(\lambda_k) - \sum_{k=1}^{N-1} f(\widehat{\lambda}_k) \quad \text{and} \quad \widetilde{f}_N := \sum_{k=1}^N f(\lambda_k - h_{11}) - \sum_{k=1}^{N-1} f(\widehat{\lambda}_k - h_{11})$$

converge in probability to

$$\Omega_f := \frac{1}{\pi} \int_{-2}^2 \frac{f(x)}{\sqrt{4-x^2}} dx.$$

More precisely, for any fixed $\epsilon > 0$,

$$\mathbf{E} f_N = \Omega_f + \mathcal{O}(N^{-2/3+\epsilon}), \quad \mathbf{E} \widetilde{f}_N = \Omega_f + \mathcal{O}(N^{-2/3+\epsilon}), \quad (6)$$

and f_N and \widetilde{f}_N fluctuate on scale $N^{-1/2}$, i.e.,

$$\mathbf{E} \left[\sqrt{N}(f_N - \mathbf{E} f_N) \right]^2 = V_f + \mathcal{O}(N^{-1/6+\epsilon}) := V_{f,1} + |\sigma_2|^2 V_{\sigma_2} + (\sigma_4 - 1)V_{f,2} + s_{11}V_{f,3} + \mathcal{O}(N^{-1/6+\epsilon}),$$

and the same relation holds for \widetilde{f}_N with constants V_f and $V_{f,i}$ replaced by \widetilde{V}_f and $\widetilde{V}_{f,i}$. The limiting variances are positive and they are given by

$$V_{f,1} = \widetilde{V}_{f,1} = \int_{-2}^2 f'(x)^2 \rho(x) dx - \left(\int_{-2}^2 x f'(x) \rho(x) dx \right)^2 - \left(\int_{-2}^2 f'(x) \rho(x) dx \right)^2, \quad (7)$$

$$V_{f,2} = \widetilde{V}_{f,2} = \left(\int_{-2}^2 x f'(x) \rho(x) dx \right)^2, \quad (8)$$

$$V_{f,3} = \left(\int_{-2}^2 f'(x) \rho(x) dx \right)^2 \quad \text{and} \quad \widetilde{V}_{f,3} = \left(\int_{-2}^2 \frac{x f''(x)}{2} \rho(x) dx \right)^2, \quad (9)$$

where $\rho(x) := \frac{1}{2\pi} \sqrt{4-x^2}$ is the density of the semicircle law and V_{σ_2} , as defined in eq. (51), is a correction term only needed when $\sigma_2 \neq 0$. For the special case of symmetric Wigner matrices H where $\sigma_2 = 1$ holds automatically, we have $V_{\sigma_2} = V_{f,1}$.

Since the error $\mathcal{O}(N^{-2/3+\epsilon})$ in (6) is of lower order than the fluctuation, the theorem also shows trivially that

$$\mathbf{E} \left[\sqrt{N}(f_N - \Omega_f) \right]^2 = V_f + \mathcal{O}(N^{-1/6+\epsilon}), \quad \mathbf{E} \left[\sqrt{N}(\widetilde{f}_N - \Omega_f) \right]^2 = \widetilde{V}_f + \mathcal{O}(N^{-1/6+\epsilon}).$$

By polarization identity, the limiting covariances of $\sqrt{N}[f_N - \Omega_f]$ and $\sqrt{N}[g_N - \Omega_g]$ may be obtained for any pair of functions f, g . In particular, Theorem 1 extends to complex test functions f by considering its real and imaginary parts separately. We also note that the condition $f \in H^2$ is not essential. The theorem holds for any $f \in H^1$, provided that $\int_{-2}^2 |\rho'(x) x f'(x)| dx < \infty$, from which we can recover the finiteness of $\widetilde{V}_{f,3}$ in (9) via an integration by parts. Finally, we remark that the same statement holds for generalized Wigner matrices where we assume $s_{ij} = 1$ only for $i = 1$ and $j > 1$. For $i \geq 2$ we only need to assume

$$\sum_{j \geq 2} s_{ij} = N - 1, \quad \max_{i,j} s_{ij} \leq C \quad (10)$$

for some constant C . We leave it to the reader to check that our proof carries over with minor modifications to this general case, as well.

Apart from the covariance structure we also identify the distribution of the fluctuations:

Theorem 2. *Under the conditions of Theorem 1 we have, for any $\epsilon > 0$, that*

$$\mathbf{E} \left[\sqrt{N}(f_N - \mathbf{E} f_N) \right]^k = (k-1)!! V_f^{k/2} + \mathcal{O}(N^{-1/6+\epsilon})$$

for even k and

$$\mathbf{E} \left[\sqrt{N}(f_N - \mathbf{E} f_N) \right]^k = \mathcal{O}(N^{-1/6+\epsilon})$$

for odd k if and only if h_{11} is normally distributed. In particular, $\sqrt{N}(f_N - \mathbf{E} f_N)$ converges in distribution to a centered normal distribution of variance V_f if and only if h_{11} is normally distributed. The same statement holds true with f_N and V_f being replaced by \widetilde{f}_N and \widetilde{V}_f .

Applied to rectangular Young diagrams, this result translates to:

Theorem 3. *Let the Wigner matrix H satisfy (5), $s_{ij} = 1$ for $(i, j) \neq (1, 1)$, $\mathbf{E} |h_{1j}|^4 = \sigma_4/N^2$ for $j = 2, \dots, N$ and $\mathbf{E} h_{ij}^2 = \sigma_2/N$ for $i < j$. Then for any $\epsilon > 0$ we have*

$$\mathbf{E} w_N(E) = \Omega(E) + \mathcal{O}\left(N^{-2/3+\epsilon}\right) \quad \text{and} \quad \mathbf{E} \tilde{w}_N(E) = \Omega(E) + \mathcal{O}\left(N^{-2/3+\epsilon}\right) \quad (11)$$

uniformly for any $E \in \mathbb{R}$. Moreover, $w_N(E)$ and $\tilde{w}_N(E)$ fluctuate on scale $N^{-1/2}$, i.e.,

$$\mathbf{E} \left[\sqrt{N} (w_N(E) - \mathbf{E} w_N(E)) \right]^2 = V(E) + \mathcal{O}\left(N^{-1/6+\epsilon}\right) := V_1(E) + |\sigma_2|^2 V_{\sigma_2}(E) + (\sigma_4 - 1) V_2(E) + s_{11} V_3(E)$$

and the same relation holds for \tilde{w}_N with functions $V(E)$ and $V_i(E)$ replaced by $\tilde{V}(E)$ and $\tilde{V}_i(E)$. The limiting variances are positive and are given by

$$\begin{aligned} V_1(E) &= \tilde{V}_1(E) = 1 - \frac{(4 - E^2)_+^3}{9\pi^2} - \frac{\left(E\sqrt{(4 - E^2)_+} + 4 \arcsin E/2\right)^2}{4\pi^2} \\ V_2(E) &= \tilde{V}_2(E) = \frac{(4 - E^2)_+^3}{9\pi^2} \\ V_3(E) &= \frac{\left(E\sqrt{(4 - E^2)_+} + 4 \arcsin E/2\right)^2}{4\pi^2} \quad \text{and} \quad \tilde{V}_3(E) = \frac{(4 - E^2)_+ E^2}{4\pi^2}, \end{aligned}$$

where it is understood that $\arcsin(\pm x) = \pm\pi/2$ for $x > 2$. The correction term $V_{\sigma_2}(E)$, that is only needed when $\sigma_2 \neq 0$, can be obtained via the general formula for V_{σ_2} from (51). For the special case of real symmetric H , we have $V_{\sigma_2}(E) = V_1(E)$. All error terms are uniform in E . Moreover, the fluctuations are Gaussian if and only if h_{11} is Gaussian.

Since the error (11) is smaller than the scale of the fluctuation, the theorem also trivially shows that

$$\mathbf{E} \left[\sqrt{N} (w_N(E) - \Omega(E)) \right]^2 = V(E) + \mathcal{O}\left(N^{-1/6+\epsilon}\right), \quad \mathbf{E} \left[\sqrt{N} (\tilde{w}_N(E) - \Omega(E)) \right]^2 = \tilde{V}(E) + \mathcal{O}\left(N^{-1/6+\epsilon}\right).$$

A simple inspection also shows that $0 \leq \tilde{V}(E) \leq V(E)$, in particular the shifted curve \tilde{w}_N not only becomes deterministic for $|E| \geq 2$, but it has smaller fluctuation than w_N everywhere. Furthermore, we have $\tilde{V}(E) > 0$ for any $|E| < 2$, while $\tilde{V}(E) = 0$ but $V(E) > 0$ for $|E| \geq 2$.

3. VARIANCE COMPUTATION

In this section we prove Theorem 1. We first introduce a commonly used (see, e.g., [14]) notion of high-probability bound which helps in keeping the notation compact.

Definition 4 (Stochastic Domination). *If*

$$X = \left(X^{(N)}(u) \mid N \in \mathbb{N}, u \in U^{(N)} \right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) \mid N \in \mathbb{N}, u \in U^{(N)} \right)$$

are families of random variables indexed by N , and possibly some parameter u , then we say that X is stochastically dominated by Y , if for all $\epsilon, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P} \left[X^{(N)}(u) > N^\epsilon Y^{(N)}(u) \right] \leq N^{-D}$$

for large enough $N \geq N_0(\epsilon, D)$. In this case we use the notation $X \prec Y$. Moreover, if we have $|X| \prec Y$, we also write $X = \mathcal{O}_\prec(Y)$.

It can be checked (see [14, Lemma 4.4]) that \prec satisfies the usual arithmetic properties, e.g. if $X_1 \prec Y_1$ and $X_2 \prec Y_2$, then also $X_1 + X_2 \prec Y_1 + Y_2$ and $X_1 X_2 \prec Y_1 Y_2$. We will say that a (sequence of) events $A = A^{(N)}$ holds with *overwhelming probability* if $\mathbf{P}(A^{(N)}) \geq 1 - N^{-D}$ for any $D > 0$ and $N \geq N_0(D)$. In particular, under the conditions (5), we have $h_{ij} \prec N^{-1/2}$ and $\max_k |\lambda_k| \leq 3$ with overwhelming probability.

Let $\chi: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cut-off function which is constant 1 inside $[-5, 5]$ and constant 0 outside $[-10, 10]$. Now define

$$f_\chi(x) := f(x)\chi(x)$$

and its almost analytic extension

$$f_\mathbb{C}(x + i\eta) := [f_\chi(x) + i\eta f'_\chi(x)] \chi(\eta).$$

Clearly, $f_{\mathbb{C}}$ is bounded and compactly supported. Then,

$$\partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) = \frac{1}{2} \frac{\partial}{\partial x} f_{\mathbb{C}}(x + i\eta) + \frac{i}{2} \frac{\partial}{\partial \eta} f_{\mathbb{C}}(x + i\eta) = \frac{i\eta}{2} \chi(\eta) f_{\chi}''(x) + \frac{i}{2} \chi'(\eta) [f_{\chi}(x) + i\eta f_{\chi}'(x)]$$

and we note that for small η ,

$$\partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) = \mathcal{O}(\eta) \quad \text{and} \quad \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) = \mathcal{O}(1). \quad (12)$$

For real λ we have

$$f_{\chi}(\lambda) = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{\partial_{\bar{z}} f_{\mathbb{C}}(z)}{\lambda - z} d\bar{z} \wedge dz = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta)}{\lambda - x - i\eta} dx d\eta$$

whenever $f \in C^2(\mathbb{R})$, as follows from Cauchy's Theorem. Since the left hand side of this equality is real, it suffices to integrate the real part of the integrand on the right hand side which conveniently is symmetric with respect to the real axis. Consequently,

$$f_{\chi}(\lambda) = \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{\partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta)}{\lambda - x - i\eta} d\eta dx. \quad (13)$$

Eq. (13) is commonly known as the *Helfffer-Sjöstrand formula* [19]. One can easily check that eq. (13) extends to $H^2(\mathbb{R})$ functions. The cut-off was chosen in such a way that with overwhelming probability $f(\lambda_k) = f_{\mathbb{C}}(\lambda_k)$ and $f(\lambda_k - h_{11}) = f_{\mathbb{C}}(\lambda_k - h_{11})$ and therefore eq. (13) yields

$$f_N = \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\mathbb{R}_+} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) [\text{Tr } G(x + i\eta) - \text{Tr } \widehat{G}(x + i\eta)] d\eta dx \quad (14)$$

and

$$\widetilde{f}_N = \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\mathbb{R}_+} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) [\text{Tr } G(x + h_{11} + i\eta) - \text{Tr } \widehat{G}(x + h_{11} + i\eta)] d\eta dx, \quad (15)$$

where for convenience we defined

$$H = \begin{pmatrix} h_{11} & h^* \\ h & \widehat{H} \end{pmatrix}, \quad H^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{H} \end{pmatrix}, \quad G(z) := (H - z)^{-1}, \quad \widehat{G}(z) := (\widehat{H} - z)^{-1}, \quad G^{(1)}(z) := (H^{(1)} - z)^{-1}.$$

We also introduce the short hand notations

$$\Delta_N(z) := \text{Tr } G(z) - \text{Tr } \widehat{G}(z) \quad \text{and} \quad \widetilde{\Delta}_N(z) := \text{Tr } G(z + h_{11}) - \text{Tr } \widehat{G}(z + h_{11}).$$

From the Schur complement formula we find

$$\Delta_N(z) = \frac{1 + \langle h, \widehat{G}(z)^2 h \rangle}{h_{11} - z - \langle h, \widehat{G}(z) h \rangle} \quad \text{and} \quad \widetilde{\Delta}_N(z) = \frac{1 + \langle h, \widehat{G}(z + h_{11})^2 h \rangle}{-z - \langle h, \widehat{G}(z + h_{11}) h \rangle}. \quad (16)$$

The basic strategy now is that we identify the leading order behavior of these two expressions and then handle the fluctuations separately. To do so, we firstly exclude a critical area very close to the real line. Since

$$\left| \eta + \eta \langle h, \widehat{G}(x + i\eta)^2 h \rangle \right| \leq \eta + \Im \langle h, \widehat{G}(x + i\eta) h \rangle \leq \left| x_0 + x + i\eta + \langle h, \widehat{G}(x + i\eta) h \rangle \right|$$

for any $x_0 \in \mathbb{R}$ we find that

$$|\eta \Delta_N(x + i\eta)| \leq 1 \quad \text{and} \quad |\eta \widetilde{\Delta}_N(x + i\eta)| \leq 1$$

for all $\eta > 0$. Therefore we can restrict our integrations in (14)–(15) to the domain $\Im z > \eta_0 := N^{-2/3}$ and find that

$$f_N = \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) \Delta_N(x + i\eta) d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right)$$

and

$$\widetilde{f}_N = \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) \widetilde{\Delta}_N(x + i\eta) d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right).$$

For $\Im z = \eta \geq \eta_0$ we claim that the leading order of Δ_N and $\widetilde{\Delta}_N$ is given by

$$\widehat{\Delta}_N(z) := \frac{1 + \frac{1}{N} \text{Tr } \widehat{G}(z)^2}{-z - \frac{1}{N} \text{Tr } \widehat{G}(z)}. \quad (17)$$

Accordingly, we split the proof of Theorem 1 effectively into two parts. We define

$$\widehat{\Omega}_f := \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) \widehat{\Delta}_N(x + i\eta) d\eta dx \quad (18)$$

and

$$F_N := \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) \left[\Delta_N(x + i\eta) - \widehat{\Delta}_N(x + i\eta) \right] d\eta dx \quad (19)$$

$$\widetilde{F}_N := \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) \left[\widetilde{\Delta}_N(x + i\eta) - \widehat{\Delta}_N(x + i\eta) \right] d\eta dx, \quad (20)$$

so that

$$f_N = \widehat{\Omega}_f + F_N + \mathcal{O}_{\prec} \left(N^{-2/3} \right) \quad \text{and} \quad \widetilde{f}_N = \widehat{\Omega}_f + \widetilde{F}_N + \mathcal{O}_{\prec} \left(N^{-2/3} \right).$$

Proposition 5 (Leading Order). *Under the assumptions of Theorem 1 we have that*

$$\widehat{\Omega}_f = \Omega_f + \mathcal{O}_{\prec} \left(N^{-2/3} \right).$$

Proposition 6 (Fluctuations). *Under the assumptions of Theorem 1 we have that*

$$\mathbf{E} F_N^2 = \frac{1}{N} V_f + \mathcal{O}_{\prec} \left(N^{-7/6} \right) \quad \text{and} \quad \mathbf{E} \widetilde{F}_N^2 = \frac{1}{N} \widetilde{V}_f + \mathcal{O}_{\prec} \left(N^{-7/6} \right).$$

Note that the error terms in these propositions are deterministic and hence could also be written as $\mathcal{O} \left(N^{-2/3+\epsilon} \right)$ or $\mathcal{O} \left(N^{-7/6+\epsilon} \right)$ for any $\epsilon > 0$, respectively, but for simplicity we keep the $\mathcal{O}_{\prec}(\dots)$ notation for deterministic quantities as well.

These two propositions are clearly sufficient for proving Theorem 1. The positivity of V_f and \widetilde{V}_f defined (7)–(9) follows from $1 = \sigma_2 \leq \sigma_4$ and from simple Schwarz inequalities

$$\begin{aligned} \left(\int_{-2}^2 x \rho(x) f'(x) dx \right)^2 &\leq \int_{-2}^2 \rho(x) f'(x)^2 dx, \\ \left(\int_{-2}^2 x \rho(x) \left(f'(x) - \int \rho f' \right) dx \right)^2 &\leq \left(\int_{-2}^2 \rho(x) \left(f(x) - \int \rho f \right)^2 \rho(x) dx \right), \end{aligned}$$

using that the semicircle density ρ is symmetric and $\int x^2 \rho(x) dx = 1$.

3.1. Leading Order Integral. This section is devoted to the proof of Proposition 5. We rely on the local semicircle law in the averaged form (see [15] or [14, Theorem 2.3])

$$m_N(z) := \frac{1}{N} \text{Tr} \widehat{G}(z) = m(z) + \mathcal{O}_{\prec} \left(\frac{1}{N\eta} \right) \quad (21)$$

and the entry-wise form

$$\widehat{G}_{ij}(z) - \delta_{ij} m(z) \prec \frac{1}{\sqrt{N\eta}} \quad (22)$$

which holds true for all $\eta = \Im z > \eta_0$. Here $m(z)$ is the Stieltjes transform of the semicircle distribution, i.e.,

$$m(z) := \int_{-2}^2 \frac{1}{x - z} \rho(x) dx = \frac{1}{2\pi} \int_{-2}^2 \frac{\sqrt{4 - x^2}}{x - z} dx = \frac{-z + \sqrt{z^2 - 4}}{2},$$

where we chose the branch of the square root with positive imaginary part. Note that \widehat{G} is an $(N - 1) \times (N - 1)$ matrix but we still normalize its trace by $1/N$ to define m_N ; this unconventional notation will simplify some formulas later. Strictly speaking, the sum of the variances in each row of \widehat{H} is not exactly one as required in [15, 14], partly due to the removal of one column and partly due to the relaxed bound $\mathbf{E} |h_{ii}|^2 \leq C/N$ on the diagonal elements. Nevertheless, we still have $\sum_{i=2}^N \mathbf{E} |h_{ij}|^2 = 1 + \mathcal{O}(N^{-1})$ for each $j = 2, 3, \dots, N$ and it is easy to check that the proof of [14, Theorem 2.3] goes through without any essential changes.

Thus

$$\widehat{\Delta}_N(z) = \frac{1 + \frac{1}{N} \text{Tr} \widehat{G}(z)^2}{-z - m(z)} + \mathcal{O}_{\prec} \left(\frac{1}{N\eta} \right) = m(z) \left[1 + \frac{1}{N} \text{Tr} \widehat{G}(z)^2 \right] + \mathcal{O}_{\prec} \left(\frac{1}{N\eta} \right),$$

where we used the relation $m(z) = 1/(-z - m(z))$. Since $\partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta) = \mathcal{O}(\eta)$ for small η the error term, when inserted in (18) only gives a contribution of $1/N$. Thus eq. (18) becomes

$$\widehat{\Omega}_f = \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(z) m(z) \left[1 + \frac{1}{N} \text{Tr} \widehat{G}(z)^2 \right] d\eta dx + \mathcal{O}_{\prec}(N^{-1}),$$

where from now on we shall always use the shorthand notation $z = x + i\eta$. Noting that

$$1 + \frac{1}{N} \text{Tr} \widehat{G}(x + i\eta)^2 = \partial_{\eta} \left[\eta - i \frac{1}{N} \text{Tr} \widehat{G}(x + i\eta) \right] = \partial_{\eta} [\eta - i m_N(x + i\eta)],$$

and $-i\partial_{\eta} h(z) = \partial_z h(z)$ for analytic h , we can now perform an integration by parts to find

$$\begin{aligned} \widehat{\Omega}_f &= \frac{2}{\pi} \Re \int_{\mathbb{R}} \partial_{\bar{z}} f_{\mathbb{C}}(z_0) m(z_0) [\eta - i m_N(z_0)] dx - \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\eta} (\partial_{\bar{z}} f_{\mathbb{C}}(z) m(z)) [\eta - i m_N(z)] d\eta dx + \mathcal{O}_{\prec}(N^{-1}) \\ &= \frac{2}{\pi} \Re \int_{\mathbb{R}} \partial_{\bar{z}} f_{\mathbb{C}}(z_0) m(z_0) [\eta - i m(z_0)] dx - \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\eta} (\partial_{\bar{z}} f_{\mathbb{C}}(z) m(z)) [\eta - i m(z)] d\eta dx + \mathcal{O}_{\prec}(N^{-1}) \\ &\quad + \frac{2}{\pi} \Re \int_{\mathbb{R}} \partial_{\bar{z}} f_{\mathbb{C}}(z_0) m(z_0) i [m_N(z_0) - m(z_0)] dx - \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\eta} (\partial_{\bar{z}} f_{\mathbb{C}}(z) m(z)) i [m_N(z) - m(z)] d\eta dx \\ &= \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(z) m(z) [1 + m'(z)] d\eta dx + \mathcal{O}_{\prec}(N^{-1}) + \mathcal{O}_{\prec}(N^{-1} |\log \eta_0|) \end{aligned}$$

where we used that $\partial_{\bar{z}} f_{\mathbb{C}}(x + i\eta)$ scales like η near the real axis and the local semicircle law from eq. (21). For the main term we need the following simple lemma.

Lemma 7. *Let $\phi, \psi: [-10, 10] \times [0, 10i] \rightarrow \mathbb{C}$ be functions such that $\partial_{\bar{z}} \psi(z) \equiv 0$, $\phi, \psi \in H^1$ and ϕ vanishes at the left, right and top boundary of the integration region. Then for any $\eta_0 \in [0, 10]$, we have*

$$\int_{-10}^{10} \int_{\eta_0}^{10} [\partial_{\bar{z}} \phi(z)] \psi(z) d\eta dx = \frac{1}{2i} \int_{-10}^{10} \phi(x + i\eta_0) \psi(x + i\eta_0) dx, \quad z = x + i\eta.$$

Proof. This follows from the computation

$$\begin{aligned} \int_{-10}^{10} \int_{\eta_0}^{10} [\partial_{\bar{z}} \phi(z)] \psi(z) d\eta dx &= \frac{1}{2i} \int_{-10}^{10} \int_{\eta_0}^{10} [\partial_{\bar{z}} \phi(z)] \psi(z) d\bar{z} \wedge dz = \frac{1}{2i} \int_{-10}^{10} \int_{\eta_0}^{10} d(\phi(z) \psi(z) dz) \\ &= \frac{1}{2i} \int_{-10}^{10} \phi(x + i\eta_0) \psi(x + i\eta_0) dx, \end{aligned}$$

where we used Stokes' Theorem in the ultimate step. \square

We apply this together with $\Im m(x)[1 + m'(x)] = (4 - x^2)^{-1/2}$ and (12) to extend the integration to the real axis and conclude that

$$\widehat{\Omega}_f = \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(z) \widehat{\Delta}_N(z) d\eta dx = \Omega_f + \mathcal{O}_{\prec}(N^{-2/3}) = \frac{1}{\pi} \int_{-2}^2 \frac{f(x)}{\sqrt{4 - x^2}} dx + \mathcal{O}_{\prec}(N^{-2/3}),$$

completing the proof of Proposition 5.

3.2. Fluctuation Integral. We now turn to the proof of Proposition 6. We formulate the main estimate as a lemma:

Lemma 8. *For any $\eta > \eta_0$ we have that*

$$\Delta_N(z) - \widehat{\Delta}_N(z) = \partial_z \frac{\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11}}{-z - m_N(z)} + \mathcal{O}_{\prec} \left(\frac{1}{N\eta^2} \right) \quad (23)$$

and

$$\widetilde{\Delta}_N(z) - \widehat{\Delta}_N(z) = \partial_z \frac{\langle h, \widehat{G}(z + h_{11}) h \rangle - m_N(z)}{-z - m_N(z)} + \mathcal{O}_{\prec} \left(\frac{1}{N\eta^2} \right). \quad (24)$$

Proof. This lemma relies on the following large deviation bound (see, e.g. [14, Theorem C.1])

$$\langle h, Ah \rangle = \frac{1}{N} \operatorname{Tr} A + \mathcal{O}_{\prec} \left(\frac{1}{N} \sqrt{\operatorname{Tr} |A|^2} \right). \quad (25)$$

To prove eq. (23) we write the difference $\Delta_N - \widehat{\Delta}_N$ from (16) and (17) as

$$\Delta_N(z) - \widehat{\Delta}_N(z) = \frac{(-z - m_N(z)) \left(\langle h, \widehat{G}(z)^2 h \rangle - m'_N(z) \right) - (-1 - m'_N(z)) \left(\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11} \right)}{(-z - m_N(z))^2 - (-z - m_N(z)) \left(\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11} \right)}.$$

Now it follows from eq. (25) and (21) that

$$\langle h, \widehat{G}(z) h \rangle - m_N(z) \prec \frac{1}{N} \sqrt{\operatorname{Tr} |\widehat{G}(z)|^2} \leq \frac{1}{N} \sqrt{\frac{1}{\eta} \Im \operatorname{Tr} \widehat{G}(z)} \prec \frac{1}{\sqrt{N\eta}} \quad (26)$$

and also

$$\langle h, \widehat{G}(z)^2 h \rangle - m'_N(z) \prec \frac{1}{N} \sqrt{\operatorname{Tr} |\widehat{G}(z)|^4} \leq \frac{1}{N\eta} \sqrt{\operatorname{Tr} |\widehat{G}(z)|^2} \prec \frac{1}{\sqrt{N\eta^3}}. \quad (27)$$

We can therefore conclude that $\Delta_N(z) - \widehat{\Delta}_N(z)$ can be estimated as

$$\begin{aligned} & \frac{(-z - m_N(z)) \left(\langle h, \widehat{G}(z)^2 h \rangle - m'_N(z) \right) - (-1 - m'_N(z)) \left(\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11} \right)}{(-z - m_N(z))^2} + \mathcal{O}_{\prec} \left(\frac{1}{N\eta^2} \right) \\ &= \partial_z \frac{\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11}}{-z - m_N(z)} + \mathcal{O}_{\prec} \left(\frac{1}{N\eta^2} \right). \end{aligned}$$

The proof of eq. (24) is identical and shall be omitted. \square

We now use eq. (23) to start estimating the fluctuations F_N of f_N as defined in eq. (19) via an integration by parts (with $z_0 = x + i\eta_0$)

$$\begin{aligned} F_N &= -\frac{2}{\pi} \Re \int_{\mathbb{R}} \partial_{\bar{z}} f_{\mathbb{C}}(z_0) i \frac{\langle h, \widehat{G}(z_0) h \rangle - m_N(z_0) - h_{11}}{-z_0 - m_N(z_0)} dx \\ &\quad + \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) i \frac{\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11}}{-z - m_N(z)} d\eta dx + \mathcal{O}_{\prec} \left(\frac{-\log \eta_0}{N} \right) \end{aligned}$$

and continue with the estimate

$$\frac{\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11}}{-z - m_N(z)} = \frac{\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11}}{-z - m(z)} + \mathcal{O}_{\prec} \left(\frac{1}{(N\eta)^{3/2}} \right)$$

from (26) and (21) to find that

$$\begin{aligned} F_N &= -\frac{2}{\pi} \Re \int_{\mathbb{R}} m(z_0) \partial_{\bar{z}} f_{\mathbb{C}}(z_0) i \left[\langle h, \widehat{G}(z_0) h \rangle - m_N(z_0) - h_{11} \right] dx \\ &\quad + \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) i \left[\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11} \right] d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right) \\ &= -\frac{2}{\pi} \Im \int_{\mathbb{R}} \int_{\eta_0}^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) \left[\langle h, \widehat{G}(z) h \rangle - m_N(z) - h_{11} \right] d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right), \end{aligned} \quad (28)$$

where we used in the last step that

$$\left| \partial_{\bar{z}} f_{\mathbb{C}}(z_0) \left[\langle h, \widehat{G}(z_0) h \rangle - m_N(z_0) - h_{11} \right] \right| \prec \sqrt{\frac{\eta_0}{N}} \leq N^{-2/3}$$

from (26) and (12). Similarly one finds that

$$\begin{aligned}
\tilde{F}_N &:= \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} \partial_{\bar{z}} f_{\mathbb{C}}(z) [\tilde{\Delta}_N(z) - \hat{\Delta}_N(z)] d\eta dx \\
&= \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) i \left[\langle h, \hat{G}(z + h_{11}) h \rangle - m_N(z) \right] d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right) \\
&= \frac{2}{\pi} \Re \int_{\mathbb{R}} \int_{\eta_0}^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) i \left[\langle h, \hat{G}(z + h_{11}) h \rangle - m_N(z + h_{11}) + m(z + h_{11}) - m(z) \right] d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right) \\
&= -\frac{2}{\pi} \Im \int_{\mathbb{R}} \int_{\eta_0}^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) \left[\langle h, \hat{G}(z + h_{11}) h \rangle - m_N(z + h_{11}) + h_{11} m'(z) \right] d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right) \quad (29)
\end{aligned}$$

where in the penultimate step we used the local semicircle law (21) and integrated the error term $(N\eta)^{-1}$ at an expense of $N^{-1} |\log \eta_0|$ and in the last step estimated

$$m(z + h_{11}) - m(z) = h_{11} m'(z) + \mathcal{O}_{\prec} \left(\frac{1}{\eta^{3/2} N} \right),$$

where the error term, after integration, contributes an error of at most $N^{-2/3}$.

Both fluctuation estimates from eqs. (28) and (29) have two convenient properties: Firstly, the leading order expressions for F_N and \tilde{F}_N have zero mean and secondly, the fluctuations in them stemming from h_{11} and the ones from h and $\hat{G}(z)$ can be separated. Indeed,

$$\mathbf{E} \left[\langle h, \hat{G}(z + h_{11}) h \rangle - m_N(z + h_{11}) + h_{11} m'(z) \right]^2 = \mathbf{E} \left[\langle h, \hat{G}(z + h_{11}) h \rangle - m_N(z + h_{11}) \right]^2 + \mathbf{E} [h_{11} m'(z)]^2$$

since the expectation with respect to h , conditioned on h_{11} of the first term on the rhs. is zero and h and h_{11} are independent. Similarly,

$$\mathbf{E} \left[\langle h, \hat{G}(z) h \rangle - m_N(z) - h_{11} \right]^2 = \mathbf{E} \left[\langle h, \hat{G}(z) h \rangle - m_N(z) \right]^2 + \mathbf{E} [h_{11}]^2.$$

Therefore we can start computing the variances as

$$\begin{aligned}
\mathbf{E} F_N^2 &= \mathbf{E} \left(\frac{2}{\pi} \Im \int_{\mathbb{R}} \int_{\eta_0}^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) \left[\langle h, \hat{G}(z) h \rangle - m_N(z) \right] d\eta dx \right)^2 \\
&\quad + \frac{s_{11}}{N} \left(\frac{2}{\pi} \Im \int_{\mathbb{R}} \int_0^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) d\eta dx \right)^2 + \mathcal{O}_{\prec} \left(N^{-7/6} \right) \quad (30)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E} \tilde{F}_N^2 &= \mathbf{E} \left(\frac{2}{\pi} \Im \int_{\mathbb{R}} \int_{\eta_0}^{10} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) \left[\langle h, \hat{G}(z + h_{11}) h \rangle - m_N(z + h_{11}) \right] d\eta dx \right)^2 \\
&\quad + \frac{s_{11}}{N} \left(\frac{2}{\pi} \Im \int_{\mathbb{R}} \int_0^{10} m(z) m'(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z) d\eta dx \right)^2 + \mathcal{O}_{\prec} \left(N^{-7/6} \right). \quad (31)
\end{aligned}$$

Note that in the second terms we extended the integration domain of η starting from 0 instead of η_0 at a negligible error. The second terms are already deterministic and explicitly computable using Lemma 7 and they give rise to the $V_{f,3}$ and $\tilde{V}_{f,3}$ coefficients in (9). When taking expectations, we frequently use the property that if $X = \mathcal{O}_{\prec}(Y)$, $Y \geq 0$ and $|X| \leq N^C$ for some constant C , then $\mathbf{E} |X| \prec \mathbf{E} Y$, or, equivalently, $\mathbf{E} |X| \leq N^{\epsilon} \mathbf{E} Y$ for any $\epsilon > 0$ and $N \geq N_0(\epsilon)$.

For the first term we introduce short-hand notations

$$g(z) := \frac{2}{\pi} m(z) \partial_{\eta} \partial_{\bar{z}} f_{\mathbb{C}}(z), \quad X(z) := \sqrt{N} \left[\langle h, \hat{G}(z) h \rangle - m_N(z) \right] \quad (32)$$

to write

$$F'_N := \frac{1}{\sqrt{N}} \mathbf{E} \left(\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) X(z) d\eta dx \right)^2, \quad \tilde{F}'_N := \frac{1}{\sqrt{N}} \mathbf{E} \left(\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) X(z + h_{11}) d\eta dx \right)^2.$$

For complex numbers z, w we can expand

$$(\Im z)(\Im w) = \frac{1}{2} \Re [\bar{z}w - zw] \quad (33)$$

to write out

$$F'_N = \frac{1}{N} \frac{1}{2} \Re \iint_{\mathbb{R}} \iint_{\eta_0}^{10} [g(z)g(\bar{z}') \mathbf{E} X(z)X(\bar{z}') - g(z)g(z') \mathbf{E} X(z)X(z')] d\eta d\eta' dx dx' \quad (34)$$

where we used that $\overline{X(z)} = X(\bar{z})$ and $\overline{g(z)} = g(\bar{z})$. To work out the expectations, we expand

$$X(z)X(z') = N \left(\sum_{i \neq j} \bar{h}_i G_{ij} h_j + \sum_i \left[|h_i|^2 - \frac{1}{N} \right] G_{ii} \right) \left(\sum_{l \neq k} \bar{h}_l G'_{lk} h_k + \sum_l \left[|h_l|^2 - \frac{1}{N} \right] G'_{ll} \right)$$

where we introduced the shorthand notations

$$G = \widehat{G}(z), \quad G' = \widehat{G}(z').$$

Note that we have redefined the notation G but it should not create any confusion since the full resolvent matrix $G(z)$ will not appear any more in the rest of the paper. To keep the notation simple we generally index the $(N-1) \times (N-1)$ matrices G, G' and the $(N-1)$ vector h by integers $\{2, \dots, N\}$. In particular, all sums involving G and G' run from 2 to N if not stated otherwise. We then compute the expectation $\mathbf{E}_1 = \mathbf{E}(\cdot | H^{(1)})$ conditioned on $H^{(1)}$ to find

$$\begin{aligned} \mathbf{E}_1[X(z)X(z')] &= N \sum_{i \neq j} \left(G_{ij} G'_{ji} \mathbf{E} |h_i|^2 |h_j|^2 + G_{ij} G'_{ij} \mathbf{E} \bar{h}_i^2 h_j^2 \right) + N \sum_i \mathbf{E} \left[|h_i|^2 - \frac{1}{N} \right]^2 G_{ii} G'_{ii} \\ &= \frac{1}{N} \sum_{i \neq j} \left(G_{ij} G'_{ji} + |\sigma_2|^2 G_{ij} G'_{ij} \right) + \frac{\sigma_4 - 1}{N} \sum_i G_{ii} G'_{ii} \\ &= \frac{1}{N} \sum_{i \neq j} \left(G_{ij} G'_{ji} + |\sigma_2|^2 G_{ij} G'_{ij} \right) + (\sigma_4 - 1) m(z) m(z') + \mathcal{O}_{\prec} \left(\frac{1}{\sqrt{N\eta}} + \frac{1}{\sqrt{N\eta'}} + \frac{1}{N\sqrt{\eta\eta'}} \right), \end{aligned} \quad (35)$$

where we recall that $\mathbf{E} h_{ij}^2 = \sigma_2/N$ for $i < j$ and $\mathbf{E} h_{ij} = \bar{\sigma}_2/N$ for $i > j$. For the computation of the first term we need a lemma:

Lemma 9. *Let $\eta, \eta' > 0$. Then for z, z' with $|\Im z| = \eta$ and $|\Im z'| = \eta'$ it holds that*

$$\frac{1}{N} \sum_{i \neq j} G_{ij} G'_{ji} = \frac{m(z)^2 m(z')^2}{1 - m(z) m(z')} + \mathcal{O}_{\prec} \left(\frac{1}{(\eta + \eta') \sqrt{N\eta\eta'}} \left[\frac{1}{\sqrt{\eta}} + \frac{1}{\sqrt{\eta'}} + \frac{1}{\sqrt{N\eta\eta'}} \right] \right) \quad (36)$$

and

$$\begin{aligned} \frac{1}{N} \sum_{i \neq j} G_{ij} G'_{ij} &= m(z) m(z') \frac{(1 + m(z) m(z') \Re \sigma_2) \frac{\tan[m(z) m(z') \Im \sigma_2]}{m(z) m(z') \Im \sigma_2} - 1}{1 - \Re \sigma_2 \frac{\tan[m(z) m(z') \Im \sigma_2]}{\Im \sigma_2}} \\ &\quad + \mathcal{O}_{\prec} \left(\frac{1}{(\eta + \eta') \sqrt{N\eta\eta'}} \left[\frac{1}{\sqrt{\eta}} + \frac{1}{\sqrt{\eta'}} + \frac{1}{\sqrt{N\eta\eta'}} \right] \right) \end{aligned} \quad (37)$$

(if $\Im \sigma_2 = 0$, then we use the convention that $\tan x/x = 0$ for $x = 0$).

We remark that the $(\eta + \eta')^{-1}$ factor in the error term can be substantially improved if $\Im z$ and $\Im z'$ has the same sign, see e.g. [13] for the special $z = z'$ case, but the same argument works in the general case.

Proof. The proof of this lemma follows the techniques used in [13]. We let $G^{(j)}$ denote the resolvent of the minor of \widehat{H} after removing the j -th row and column. We have the resolvent identity

$$G_{ij} = -G_{ii} \sum_j^{(i)} G_{ik}^{(j)} h_{kj}, \quad i \neq j,$$

where the summation runs over all $j = 2, 3, \dots, N$ except $j = i$; this exclusion is indicated with the upper index on the summation. Using the local semicircle law (21), we find that for any fixed i

$$\begin{aligned}
\frac{1}{N} \sum_j^{(i)} \mathbf{E}_j [G_{ij} G'_{ji}] &= \frac{1}{N} \sum_j^{(i)} \mathbf{E}_j \left[\frac{m(z)m(z')}{G_{jj} G'_{jj}} G_{ij} G'_{ji} \right] + \mathcal{O}_{\prec}(\Psi) \\
&= \frac{1}{N} m(z)m(z') \sum_j^{(i)} \mathbf{E}_j \left[\left(\sum_k^{(j)} G_{ik}^{(j)} h_{kj} \right) \left(\sum_l^{(j)} h_{jl} G'_{li}^{(j)} \right) \right] + \mathcal{O}_{\prec}(\Psi) \\
&= \frac{1}{N^2} m(z)m(z') \sum_j^{(i)} \sum_k^{(j)} G_{ik}^{(j)} G'_{ki}^{(j)} + \mathcal{O}_{\prec}(\Psi) \\
&= \frac{1}{N^2} m(z)m(z') \sum_j^{(i)} \left[\sum_k^{(ij)} G_{ik} G'_{ki} + G_{ii} G'_{ii} \right] + \mathcal{O}_{\prec}(\Psi) \\
&= \frac{1}{N} m(z)m(z') \left[\sum_k^{(i)} G_{ik} G'_{ki} + m(z)m(z') \right] + \mathcal{O}_{\prec}(\Psi)
\end{aligned} \tag{38}$$

where in the fourth equality we used

$$G_{ik}^{(j)} = G_{ik} - \frac{G_{ij} G_{jk}}{G_{jj}} = G_{ik} + \mathcal{O}_{\prec} \left(\frac{1}{N\eta} \right)$$

and the analogous identity for G' and we introduced the short hand notation

$$\Psi = \frac{1}{\sqrt{N^3 \eta^2 \eta'}} + \frac{1}{\sqrt{N^3 \eta \eta'^2}} + \frac{1}{N^2 \eta \eta'}$$

for the error term. We now follow the fluctuation averaging analysis from [13, Proof of Prop. 5.3 in Sections 6–7]. This proof was given for the case when the spectral parameters of the resolvents were identical, $z = z'$, but a simple inspection shows that the argument verbatim also applies to the $z \neq z'$ case. We conclude that

$$\frac{1}{N} \sum_j^{(i)} G_{ij} G'_{ji} = \frac{1}{N} \sum_j^{(i)} \mathbf{E}_j [G_{ij} G'_{ji}] + \mathcal{O}_{\prec}(\Psi). \tag{39}$$

Therefore, after summing over i we have

$$[1 - m(z)m(z')] \frac{1}{N} \sum_{j \neq i} G_{ij} G'_{ji} = m(z)^2 m(z')^2 + \mathcal{O}_{\prec}(N\Psi). \tag{40}$$

To finish the proof, we note that by an elementary calculation

$$\frac{1}{|1 - m(z)m(z')|} \leq \frac{C}{\eta + \eta'} \tag{41}$$

since

$$|m(x + i\eta)| \leq 1 - c|\eta| \tag{42}$$

and therefore

$$\frac{1}{N} \sum_{i \neq j} G_{ij} G'_{ji} = \frac{m(z)^2 m(z')^2}{1 - m(z)m(z')} + \mathcal{O}_{\prec} \left(\frac{N\Psi}{\eta + \eta'} \right).$$

This completes the proof of (36).

For the proof of eq. (37) we have to derive a vector self-consistent equation instead of the scalar one. We again start by noting that for $i \neq j$

$$\begin{aligned} \mathbf{E}_j G_{ij} G'_{ij} &= m(z) m(z') \sum_k^{(j)} G_{ik} G'_{ik} \mathbf{E} h_{kj}^2 + \mathcal{O}_{\prec}(\Psi) \\ &= m(z) m(z') \sum_k^{(i)} G_{ik} G'_{ik} \mathbf{E} h_{kj}^2 + m(z)^2 m(z')^2 \mathbf{E} h_{ij}^2 + \mathcal{O}_{\prec}(\Psi) \\ &= m(z) m(z') \sum_k^{(i)} F_{jk} \mathbf{E}_k G_{ik} G'_{ik} + m(z)^2 m(z')^2 F_{ji} + \mathcal{O}_{\prec}(\Psi), \end{aligned}$$

where we introduced the matrix F with matrix elements

$$F_{jk} := \mathbf{E} h_{kj}^2 = \frac{1}{N} \left[\mathbf{1}(k < j) \sigma_2 + \mathbf{1}(k > j) \overline{\sigma}_2 + \mathbf{1}(k = j) \right].$$

For every fixed i , we have therefore derived a self-consistent equation for the (column) vector

$$v^{(i)} = ((1 - \delta_{ij}) \mathbf{E}_j G_{ij} G'_{ij})_{j=2}^N$$

which can be written as

$$[\mathbf{1} - m(z) m(z') F] v^{(i)} = m(z)^2 m(z')^2 \left[F - \frac{1}{N} \mathbf{1} \right] e_i + \mathcal{O}_{\prec}(\Psi),$$

where $e_i = (0, 0, \dots, 1, \dots, 0)^T$ is the standard i -th basis vector of \mathbb{C}^{N-1} . To invert this equation while controlling the error term, we have estimate

$$\left\| [\mathbf{1} - m(z) m(z') F]^{-1} \right\|_{\ell^\infty \rightarrow \ell^\infty}.$$

To do so, we first note that

$$\left\| [\mathbf{1} - m(z) m(z') F]^{-1} \right\|_{\ell^2 \rightarrow \ell^2} \leq (1 - |m(z)| |m(z')| \|F\|_{\ell^2 \rightarrow \ell^2})^{-1} \leq \frac{C}{\eta + \eta'},$$

where we used that F is Hermitian and of norm at most 1 and (42) (the norm here is induced by the usual ℓ^2 norm $\|u\|_2 := (\sum_i |u_i|^2)^{1/2}$ on \mathbb{C}^{N-1}). Next, if $(1 - m(z) m(z') F)u = v$, then

$$\|u\|_\infty \leq \|v\|_\infty + \|Fu\|_\infty \leq \frac{C}{\eta + \eta'} \|v\|_\infty,$$

where we used

$$\|Fu\|_\infty \leq \frac{1}{N} \|u\|_1 \leq \frac{1}{\sqrt{N}} \|u\|_2 = \frac{1}{\sqrt{N}} \left\| [\mathbf{1} - m(z) m(z') F]^{-1} v \right\|_2 \leq \frac{C}{\eta + \eta'} \frac{1}{\sqrt{N}} \|v\|_2 \leq \frac{C}{\eta + \eta'} \|v\|_\infty,$$

so that also

$$\left\| [\mathbf{1} - m(z) m(z') F]^{-1} \right\|_{\ell^\infty \rightarrow \ell^\infty} \leq \frac{C}{\eta + \eta'}$$

for $\eta, \eta' \leq C$. After inversion we find that

$$v^{(i)} = m(z)^2 m(z')^2 (\mathbf{1} - m(z) m(z') F)^{-1} \left(F - \frac{1}{N} \mathbf{1} \right) e_i + \mathcal{O}_{\prec} \left(\frac{\Psi}{\eta + \eta'} \right).$$

Using fluctuation averaging once more (see (39)) we can conclude that

$$\begin{aligned} \frac{1}{N} \sum_{i \neq j} G_{ij} G'_{ij} &= \frac{1}{N} \sum_{i \neq j} \mathbf{E}_j G_{ij} G'_{ij} + \mathcal{O}_{\prec}(N\Psi) \\ &= m(z)^2 m(z')^2 e^T (\mathbf{1} - m(z) m(z') F)^{-1} \left(F - \frac{1}{N} \mathbf{1} \right) e + \mathcal{O}_{\prec} \left(\frac{N\Psi}{\eta + \eta'} \right), \end{aligned} \tag{43}$$

where $e = N^{-1/2}(1, \dots, 1)^T \in \mathbb{C}^{N-1}$. We now introduce the $(N-1) \times (N-1)$ matrix

$$S := \frac{1}{N} \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ -1 & \dots & -1 & 0 \end{pmatrix}.$$

Notice that $F = \frac{1}{N}\mathbb{1} + (\Re\sigma_2)(ee^T - \frac{1}{N}\mathbb{1}) + i(\Im\sigma_2)S$. We find, through an elementary computation, that

$$\begin{aligned} m(z)m(z') \left\langle e, (\mathbb{1} - m(z)m(z')F)^{-1} \left(F - \frac{1}{N}\mathbb{1} \right) e \right\rangle \\ = \frac{(1 + m(z)m(z')\Re\sigma_2) \left\langle e, (\mathbb{1} - im(z)m(z')\Im\sigma_2 S)^{-1} e \right\rangle - 1}{1 - m(z)m(z')\Re\sigma_2 \left\langle e, (\mathbb{1} - im(z)m(z')\Im\sigma_2 S)^{-1} e \right\rangle} + \mathcal{O}_{\prec}(N^{-1}). \end{aligned}$$

It remains to compute

$$\langle e, (\mathbb{1} - \alpha S)^{-1} e \rangle = \sum_{k=0}^{\infty} \alpha^k \langle e, S^k e \rangle$$

for $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. For any vector $f \in \mathbb{C}^{N-1}$,

$$(Sf)_n = -\frac{1}{N} \sum_{n' < n} f_{n'} + \frac{1}{N} \sum_{n' > n} f_{n'} = \frac{1}{N} \sum_{n'=2}^N h_{n-n'} f_{n'}$$

where $h_k := \mathbb{1}(k < 0) - \mathbb{1}(k > 0)$. Therefore

$$\langle e, S^k e \rangle = N^{-1/2} \sum_{n=2}^N (S^k e)_n = N^{-3/2} \sum_{n, n'=2}^N h_{n-n'} (S^{k-1} e)_{n'} = \dots = N^{-k-1} \sum_{n_0, \dots, n_k=2}^N h_{n_0-n_1} \dots h_{n_{k-1}-n_k}.$$

By symmetry, $\langle e, S^k e \rangle = 0$ for odd k . Otherwise one finds via a Riemann sum approximation that

$$\langle e, S^{2k} e \rangle = \int_0^1 \dots \int_0^1 h(x_0 - x_1) \dots h(x_{2k-1} - x_{2k}) dx_0 \dots dx_{2k} + \mathcal{O}(N^{-1}),$$

where $h(x) = \mathbb{1}(x < 0) - \mathbb{1}(x > 0)$ is the Heaviside function and where we added the missing $n_i = 1$ terms at an expense of $\mathcal{O}(N^{-1})$. Via an easy induction we see that

$$\langle e, S^{2k} e \rangle = (-1)^k \frac{2^{2k}(2^{2k} - 1)}{(2k)!} B_{2k} + \mathcal{O}(N^{-1}),$$

where B_k is the k -th Bernoulli number. Consequently,

$$\langle e, (\mathbb{1} - \alpha S)^{-1} e \rangle = \frac{\tanh \alpha}{\alpha} + \mathcal{O}(N^{-1}).$$

We now use this with $\alpha = im(z)m(z')\Im\sigma_2$ to conclude that

$$m(z)m(z') \left\langle e, (\mathbb{1} - m(z)m(z')F)^{-1} \left(F - \frac{1}{N}\mathbb{1} \right) e \right\rangle = \frac{(1 + m(z)m(z')\Re\sigma_2) \frac{\tan[m(z)m(z')\Im\sigma_2]}{m(z)m(z')\Im\sigma_2} - 1}{1 - m(z)m(z')\Re\sigma_2 \frac{\tan[m(z)m(z')\Im\sigma_2]}{m(z)m(z')\Im\sigma_2}} + \mathcal{O}_{\prec}(N^{-1}).$$

Combining this with (43), we obtain

$$\frac{1}{N} \sum_{i \neq j} G_{ij} G'_{ij} = m(z)m(z') \frac{(1 + m(z)m(z')\Re\sigma_2) \frac{\tan[m(z)m(z')\Im\sigma_2]}{m(z)m(z')\Im\sigma_2} - 1}{1 - \Re\sigma_2 \frac{\tan[m(z)m(z')\Im\sigma_2]}{\Im\sigma_2}} + \mathcal{O}_{\prec} \left(\frac{N\Psi}{\eta + \eta'} \right).$$

We note that, in general, this is a finite expression since $|\Re\sigma_2| \leq \sqrt{1 - (\Im\sigma_2)^2}$ and thus in the non-trivial case where $\Re\sigma_2 \neq 0$ and $\Im\sigma_2 \neq 0$,

$$\left| \Re\sigma_2 \frac{\tan[m(z)m(z')\Im\sigma_2]}{\Im\sigma_2} \right| < \left| \sqrt{1 - (\Im\sigma_2)^2} \frac{\tan[\Im\sigma_2]}{\Im\sigma_2} \right| \leq 1. \quad \square$$

We readily check that integrating the error terms in (34) from (35) and Lemma 9 only contributes an error of magnitude $N^{-7/6}$ and conclude that if $\sigma_2 = 0$, then

$$\begin{aligned}
F'_N &= \frac{1}{2N} \Re \iint_{-10}^{10} \iint_{\eta_0} g(z)g(\bar{z}') \left[\frac{m(z)^2 m(\bar{z}')^2}{1 - m(z)m(\bar{z}')} + (\sigma_4 - 1)m(z)m(\bar{z}') \right] \\
&\quad - g(z)g(z') \left[\frac{m(z)^2 m(z')^2}{1 - m(z)m(z')} + (\sigma_4 - 1)m(z)m(z') \right] d\eta d\eta' dx dx' + \mathcal{O}_{\prec}(N^{-7/6}) \\
&= \frac{1}{2N} \Re \iint_{-10}^{10} \iint_{\eta_0} g(z)g(\bar{z}') \left[\sum_{k=2}^{\infty} [m(z)m(\bar{z}')]^k + (\sigma_4 - 1)m(z)m(\bar{z}') \right] \\
&\quad - g(z)g(z') \left[\sum_{k=2}^{\infty} [m(z)m(z')]^k + (\sigma_4 - 1)m(z)m(z') \right] d\eta d\eta' dx dx' + \mathcal{O}_{\prec}(N^{-7/6}) \\
&= \frac{1}{N} \sum_{k=2}^{\infty} \left(\Im \int_{-10}^{10} \int_{\eta_0} g(z)m(z)^k d\eta dx \right)^2 + \frac{\sigma_4 - 1}{N} \left(\Im \int_{-10}^{10} \int_{\eta_0} g(z)m(z) d\eta dx \right)^2 + \mathcal{O}_{\prec}(N^{-7/6}) \\
&= \frac{1}{N} \sum_{k=2}^{\infty} \left(\frac{1}{\pi} \Im \int_{-10}^{10} \frac{\partial_{\eta} f_{\mathbb{C}}(z_0)}{i} m(z_0)^{k+1} dx \right)^2 + \frac{\sigma_4 - 1}{N} \left(\frac{1}{\pi} \Im \int_{-10}^{10} \partial_x f_{\mathbb{C}}(z_0) m(z_0)^2 dx \right)^2 + \mathcal{O}_{\prec}(N^{-7/6}) \\
&= \frac{1}{N} \sum_{k=0}^{\infty} \left(\frac{1}{\pi} \Im \int_{-10}^{10} \frac{\partial_{\eta} f_{\mathbb{C}}(z_0)}{i} m(z_0)^{k+1} dx \right)^2 + \frac{\sigma_4 - 2}{N} \left(\frac{1}{\pi} \Im \int_{-10}^{10} f'(x) m(x)^2 dx \right)^2 \\
&\quad - \frac{1}{N} \left(\frac{1}{\pi} \Im \int_{-10}^{10} f'(x) m(x) dx \right)^2 + \mathcal{O}_{\prec}(N^{-7/6}),
\end{aligned} \tag{44}$$

where $z_0 = x + i\eta_0$ and in the penultimate step we used Lemma 7 to write

$$\Im \int_{-10}^{10} \int_{\eta_0} g(z)m(z)^k d\eta dx = \frac{2}{\pi} \Im \int_{-10}^{10} \int_{\eta_0} [\partial_{\bar{z}} \partial_{\eta} f_{\mathbb{C}}(z)] m(z)^{k+1} d\eta dx = \Im \frac{1}{i\pi} \int_{-10}^{10} \partial_{\eta} f_{\mathbb{C}}(z_0) m(z_0)^{k+1} d\eta dx$$

and that

$$\frac{\partial_{\eta} f_{\mathbb{C}}(z_0)}{i} = \partial_x f_{\mathbb{C}}(z_0) + \mathcal{O}(\eta_0) = f'(x) + \mathcal{O}(\eta_0). \tag{45}$$

Now that we reduced the area integral to a line integral, we go the geometric series steps backwards to further simplify the first term as

$$\begin{aligned}
&\frac{1}{N} \sum_{k=0}^{\infty} \left(\frac{1}{\pi} \Im \int_{-10}^{10} \frac{\partial_{\eta} f_{\mathbb{C}}(z_0)}{i} m(z_0)^{k+1} dx \right)^2 \\
&= \frac{1}{2N\pi^2} \Re \iint_{-10}^{10} \left[\left(\frac{\partial_{\eta} f_{\mathbb{C}}(z_0)}{i} \right) \overline{\left(\frac{\partial_{\eta} f_{\mathbb{C}}(z'_0)}{i} \right)} \frac{m(z_0) \overline{m(z'_0)}}{1 - m(z_0) \overline{m(z'_0)}} - \left(\frac{\partial_{\eta} f_{\mathbb{C}}(z_0)}{i} \right) \left(\frac{\partial_{\eta} f_{\mathbb{C}}(z'_0)}{i} \right) \frac{m(z_0) m(z'_0)}{1 - m(z_0) m(z'_0)} \right] dx dx'.
\end{aligned} \tag{46}$$

We would now like to approximate (46) using (45). For doing so, we have to control the error terms via the following lemma whose proof we postpone to the end of the section.

Lemma 10. *There exists an absolute constant C such that for $z_0 = x + i\eta_0$ and $z'_0 = x' + i\eta_0$ with $0 < \eta_0 \leq 1/2$ it holds that*

$$\iint_{-10}^{10} \frac{1}{|1 - m(z_0) \overline{m(z'_0)}|} dx dx' \leq C |\log \eta_0| \quad \text{and} \quad \iint_{-10}^{10} \frac{1}{|1 - m(z_0) m(z'_0)|} dx dx' \leq C |\log \eta_0|. \tag{47}$$

Using Lemma 10 and (45) we can rewrite (46) as

$$\frac{1}{2N\pi^2} \Re \iint_{\mathbb{R}} f'(x) f'(x') \left[\frac{m(z_0) \overline{m(z'_0)}}{1 - m(z_0) \overline{m(z'_0)}} - \frac{m(z_0) m(z'_0)}{1 - m(z_0) m(z'_0)} \right] dx dx' + \mathcal{O}(\eta_0 |\log \eta_0|).$$

Now, an explicit computation shows

$$\Re \left[\frac{m(z_0) \overline{m(z'_0)}}{1 - m(z_0) \overline{m(z'_0)}} - \frac{m(z_0) m(z'_0)}{1 - m(z_0) m(z'_0)} \right] = \Re \frac{-2i \Im m(z'_0)}{-x - i\eta_0 - 2\Re m(z'_0) + m(z_0)[|m(z'_0)|^2 - 1]}. \tag{48}$$

and therefore for small η_0 and (x, x') outside the square $[-2, 2]^2$ the integrand of (48) negligible. For $(x, x') \in [-2, 2]^2$ and small η_0 we have

$$\Re \frac{-2i\Im m(z'_0)}{-x - i\tilde{\eta} - 2\Re m(z'_0) + m(z_0)[|m(z'_0)|^2 - 1]} = \frac{\sqrt{4 - x'^2}\eta_0}{(x - x')^2 + \eta_0^2} + \mathcal{O}(\eta_0).$$

This expression acts like

$$\pi\sqrt{4 - x^2}\delta(x' - x)$$

for small η_0 . More formally, it is well known that for any L^2 -function h

$$\lim_{\eta \rightarrow 0} \int_{\mathbb{R}} \frac{\eta}{(x - x')^2 + \eta^2} h(x') dx' = \pi h(x)$$

in L^2 -sense. Working out an effective error term for $h \in H^1$, this allows us to conclude

$$F'_N = \frac{1}{N} \left[\int_{-2}^2 \rho(x) f'(x)^2 dx - \left(\int_{-2}^2 \rho(x) f'(x) dx \right)^2 + (\sigma_4 - 2) \left(\int_{-2}^2 \rho(x) x f'(x) dx \right)^2 \right] + \mathcal{O}_{\prec}(N^{-7/6}).$$

The computation for \tilde{F}'_N from (31), still assuming $\sigma_2 = 0$, is completely analogous and there we also have

$$\tilde{F}'_N = \frac{1}{N} \left[\int_{-2}^2 \rho(x) f'(x)^2 dx - \left(\int_{-2}^2 \rho(x) f'(x) dx \right)^2 + (\sigma_4 - 2) \left(\int_{-2}^2 \rho(x) x f'(x) dx \right)^2 \right] + \mathcal{O}_{\prec}(N^{-7/6}).$$

We can now conclude from eqs. (30) and (31) that

$$\mathbf{E} F_N^2 = F'_N + \frac{s_{11}}{N} \left(\int_{-2}^2 \rho(x) f'(x) dx \right)^2 + \mathcal{O}_{\prec}(N^{-7/6}) = \frac{V_f}{N} + \mathcal{O}_{\prec}(N^{-7/6}) \quad (49)$$

and

$$\mathbf{E} \tilde{F}_N^2 = \tilde{F}'_N + \frac{s_{11}}{N} \left(\int_{-2}^2 \rho(x) \frac{x f''(x)}{2} dx \right)^2 + \mathcal{O}_{\prec}(N^{-7/6}) = \frac{\tilde{V}_f}{N} + \mathcal{O}_{\prec}(N^{-7/6}). \quad (50)$$

So far we assumed $\sigma_2 = 0$ in (35). We now consider the general case for which we need (37) instead of (36). A similar analysis shows that we have to add an additional term $|\sigma_2|^2 V_{\sigma_2}$ to V_f and \tilde{V}_f both in (49) and (50), given by

$$V_{\sigma_2} := \frac{1}{2\pi^2} \Re \iint_{\mathbb{R}} f'(x) f'(x') \left[m(z_0)^2 \overline{m(z'_0)^2} \frac{(1 + m(z_0) \overline{m(z'_0)} \Re \sigma_2) \frac{\tan[m(z_0) \overline{m(z'_0)} \Im \sigma_2]}{\Im \sigma_2} - m(z_0) \overline{m(z'_0)}}{1 - \Re \sigma_2 \frac{\tan[m(z_0) \overline{m(z'_0)} \Im \sigma_2]}{\Im \sigma_2}} \right. \\ \left. - m(z_0)^2 m(z'_0)^2 \frac{(1 + m(z_0) m(z'_0) \Re \sigma_2) \frac{\tan[m(z_0) m(z'_0) \Im \sigma_2]}{\Im \sigma_2} - m(z_0) m(z'_0)}{1 - \Re \sigma_2 \frac{\tan[m(z_0) m(z'_0) \Im \sigma_2]}{\Im \sigma_2}} \right] dx dx'. \quad (51)$$

For the special case $\sigma_2 \in \mathbb{R}$ eq. (37) simplifies to

$$\frac{1}{N} \sum_{i \neq j} G_{ij} G'_{ij} = \frac{m(z)^2 m(z')^2 \Re \sigma_2}{1 - m(z) m(z') \Re \sigma_2} + \mathcal{O}_{\prec} \left(\frac{1}{(\eta + \eta') \sqrt{N \eta^2 \eta'}} + \frac{1}{(\eta + \eta') \sqrt{N \eta \eta'^2}} + \frac{1}{N(\eta + \eta') \eta \eta'} \right).$$

In particular, for symmetric H , where $\sigma_2 = 1$ we find that eq. (51) simplifies to $V_{\sigma_2} = V_{f,1}$. This completes the proof of Proposition 6, modulo the proof of Lemma 10.

Proof of Lemma 10. The proof of the second inequality is similar to the first one and will be left to the reader. For the first inequality, we split the integration in two regimes. We shall make use of the fact (see, e.g., [14]) that on a compact domain, say $|z_0| \leq 10$, we have

$$|1 - m(z_0)^2| \asymp \sqrt{\kappa_x + \eta} \quad \text{and} \quad \Im m(z_0) \asymp \begin{cases} \sqrt{\kappa_x + \eta_0} & \text{if } |x| \leq 2, \\ \frac{\eta_0}{\sqrt{\kappa_x + \eta_0}} & \text{else,} \end{cases} \quad (52)$$

where $\kappa_x = ||x| - 2|$ is the distance to the edge.

Firstly in the region where $\max\{|x|, |x'|\} \geq 2$, we find

$$\left| 1 - m(z_0) \overline{m(z'_0)} \right| \geq \frac{1}{2} \left[1 - |m(z_0)|^2 + 1 - |m(z'_0)|^2 \right] \geq c \sqrt{\kappa_{\max\{|x|, |x'|\}} + \eta_0},$$

where $c > 0$ is a universal constant, due to the fact that $1 - |m(z_0)|^2 = \eta_0 / \Im m(z_0)$ and (52).

Secondly, in the region where $|x|, |x'| < 2$, we write

$$1 - m(z_0)\overline{m(z'_0)} = 1 - |m(z'_0)|^2 + (m(z'_0) - m(z_0))\overline{m(z'_0)}$$

and estimate

$$\left| (m(z'_0) - m(z_0))\overline{m(z'_0)} \right| \geq c|x' - x|$$

for some positive constant c . This inequality follows from writing

$$\Re[m(z'_0) - m(z_0)] = \int_x^{x'} \Re m'(u + i\eta_0) du$$

and from the estimate

$$\Re m'(u + i\eta_0) = -\frac{2(\Im m(u + i\eta_0))^2}{|1 - m(u + i\eta_0)|^2} \leq -c$$

for $|u| \leq 2$, where we used (52) in the last step. Consequently,

$$\left| 1 - m(z_0)\overline{m(z'_0)} \right| \geq c|x - x'| - \left| 1 - |m(z'_0)|^2 \right| \geq c|x - x'| - C\frac{\eta_0}{\sqrt{\kappa_{x'} + \eta_0}}$$

and it follows that $\left| 1 - m(z_0)\overline{m(z'_0)} \right| \geq c|x - x'|/2$ whenever $|x - x'| \geq 2(C/c)\eta_0/\sqrt{\kappa_{x'} + \eta_0}$. Together with the trivial bound $\left| 1 - m(z_0)\overline{m(z'_0)} \right| \geq c\eta$ we find that the integral in (47) is bounded by $C|\log \eta_0|$. \square

4. COMPUTATION OF HIGHER MOMENTS

We now turn to the computation of higher order moments and thereby to the proof of Theorem 2. We recall from (28)–(29) that

$$F_N = -\frac{1}{\sqrt{N}}\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) \left[X(z) - \sqrt{N}h_{11} \right] d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right)$$

and

$$\tilde{F}_N = -\frac{1}{\sqrt{N}}\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) \left[X(z + h_{11}) + \sqrt{N}h_{11}m'(z) \right] d\eta dx + \mathcal{O}_{\prec} \left(N^{-2/3} \right),$$

where g and X were defined in (32). In order to compute moments of F_N and \tilde{F}_N we have to compute

$$\mathbf{E}[X(z_1) \dots X(z_k)]$$

for any $k \in \mathbb{N}$ and $z_l \in \mathbb{C} \setminus \mathbb{R}$, $l = 1, \dots, k$. We will first take the expectation with respect to the vector h in the X 's which leads to a cyclic contraction of the indices of \hat{G} . After taking the expectation with respect to \hat{H} , we will show that the leading order terms come from cycles of length two. This will effectively show that the Wick theorem holds for the random variables X . The following lemma shows that cyclic products of at least three resolvents are in fact of lower order (the same phenomenon already was observed in [13]):

Lemma 11. *For closed cycles of length $k > 2$ we have that*

$$N^{-k/2} \sum_{i_1, \dots, i_k} \mathbf{E} G_{i_1 i_2}^{(1)} \dots G_{i_{k-1} i_k}^{(k-1)} G_{i_k i_1}^{(k)} \prec \frac{1}{(\max_a \eta_a) \sqrt{N} \eta_1 \dots \eta_k} \sum_{a=1}^k \frac{1}{\sqrt{\eta_a}}, \quad (53)$$

and for open cycles of any length $k > 1$ we have that

$$N^{-(k+1)/2} \sum_{i_1, \dots, i_k} \mathbf{E} G_{i_1 i_2}^{(1)} \dots G_{i_{k-1} i_k}^{(k-1)} \prec \frac{1}{\sqrt{N} \eta_1 \dots \eta_{k-1}} \sum_{a=1}^{k-1} \frac{1}{\sqrt{\eta_a}}, \quad (54)$$

where $G^{(l)} := \hat{G}(z_l)$, $z_l \in \mathbb{C} \setminus \mathbb{R}$ with $\eta_l = |\Im z_l|$ for $l = 1, \dots, k$ and \sum indicates that the sum is performed over pairwise distinct indices. Moreover, the same bound holds true when any of the $G^{(l)}$ are replaced by their transposes or Hermitian conjugates.

Proof. We first prove eq. (53) and assume a real symmetric H . To do so, we let $\epsilon > 0$ be arbitrary and will actually prove

$$N^{-k/2} \sum_{i_1, \dots, i_k} \mathbf{E} G_{i_1 i_2}^{(1)} \dots G_{i_{k-1} i_k}^{(k-1)} G_{i_k i_1}^{(k)} \prec \frac{N^\epsilon}{(\eta_1 + \eta_k) \sqrt{N} \eta_1 \dots \eta_k} \sum_{a=1}^k \frac{1}{\sqrt{\eta_a}},$$

from which (53) follows due to the definition of \prec in Definition 4. We make use of the resolvent identity $G^{(1)} = \hat{H}G^{(1)}/z_1 - 1/z_1$ to write

$$N^{-k/2} \sum_{i_1, \dots, i_k}^{\sim} \mathbf{E} G_{i_1 i_2}^{(1)} \dots G_{i_k i_1}^{(k)} = \frac{1}{N^{k/2} z_1} \sum_{i_1, \dots, i_k}^{\sim} \sum_n \mathbf{E} h_{i_1 n} G_{n i_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)}. \quad (55)$$

We use the standard cumulant expansion (introduced in the context of random matrices in [29]) up to the third order term with a truncation

$$\mathbf{E} h f(h) = \mathbf{E} h \mathbf{E} f(h) + \mathbf{E} h^2 \mathbf{E} f'(h) + \mathcal{O} \left(\mathbf{E} \left| h^3 \mathbb{1}(|h| > N^{\tau-1/2}) \right| \|f''\|_{\infty} \right) + \mathcal{O} \left(\mathbf{E} |h|^3 \sup_{|x| \leq N^{\tau-1/2}} |f''(x)| \right), \quad (56)$$

where f is any smooth function of a real random variable h , such that the expectations exist and $\tau > 0$ is arbitrary (for a recent similar use of this formula with truncation see [18, Lemma 3.1]). This yields

$$\begin{aligned} \mathbf{E} h_{i_1 n} G_{n i_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} &= \frac{1}{N} \mathbf{E} \frac{\partial [G_{n i_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)}]}{\partial h_{i_1 n}} + R \\ &= \frac{1}{N} \mathbf{E} \frac{\partial G_{n i_2}^{(1)}}{\partial h_{i_1 n}} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} + \frac{1}{N} \sum_{a=2}^k \mathbf{E} \frac{\partial G_{i_a i_{a+1}}^{(a)}}{\partial h_{i_1 n}} G_{n i_2}^{(1)} \prod_{a \neq b=2}^k G_{i_b i_{b+1}}^{(b)} + R, \end{aligned} \quad (57)$$

where it is understood that $i_{k+1} = i_1$ and R is the error term resulting from the cumulant expansion. Using the identity

$$\frac{\partial G_{ij}}{\partial h_{kl}} = -(G_{ik} G_{lj} + G_{il} G_{kj}) / (1 + \delta_{kl}),$$

and the local law (22), the first term on the rhs. of eq. (57) becomes

$$-(G_{n i_1}^{(1)} G_{n i_2}^{(1)} + G_{n n}^{(1)} G_{i_1 i_2}^{(1)}) G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} = -m(z_1) G_{i_1 i_2}^{(1)} \dots G_{i_k i_1}^{(k)} + \mathcal{O}_{\prec} \left(\frac{1}{N^{k/2+1/2} \sqrt{\eta \eta_1}} \right),$$

whenever $n \neq i_1, i_2$ and where $\eta := \eta_1 \dots \eta_k$. If $n = i_1$ or $n = i_2$, we shall make use of the trivial estimate

$$-(G_{n i_1}^{(1)} G_{n i_2}^{(1)} + G_{n n}^{(1)} G_{i_1 i_2}^{(1)}) G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} \prec \frac{1}{N^{k/2} \sqrt{\eta}}.$$

The $a = k$ summand of the second term in eq. (57) becomes

$$-(G_{i_k i_1}^{(k)} G_{n i_1}^{(k)} + G_{i_k n}^{(k)} G_{i_1 i_1}^{(k)}) G_{n i_2}^{(1)} \dots G_{i_{k-1} i_k}^{(k-1)} = -m(z_k) G_{n i_2}^{(1)} \dots G_{i_k n}^{(k)} + \mathcal{O}_{\prec} \left(\frac{1}{N^{k/2+1/2} \sqrt{\eta \eta_k}} \right) \quad (58)$$

whenever $n \neq i_1, i_k$. For these exceptional n we shall again use the trivial $N^{-k/2} \eta^{-1/2}$ estimate. For $a \neq k$ the summand in the second term of eq. (57) can always be estimated by

$$-(G_{i_a i_1}^{(a)} G_{n i_{a+1}}^{(a)} + G_{i_a n}^{(a)} G_{i_1 i_{a+1}}^{(a)}) G_{n i_2}^{(1)} \prod_{a \neq b=2}^k G_{i_b i_{b+1}}^{(b)} \prec \frac{1}{N^{k/2} \sqrt{\eta}}$$

and this bound can be improved to

$$-(G_{i_a i_1}^{(a)} G_{n i_{a+1}}^{(a)} + G_{i_a n}^{(a)} G_{i_1 i_{a+1}}^{(a)}) G_{n i_2}^{(1)} \prod_{a \neq b=2}^k G_{i_b i_{b+1}}^{(b)} \prec \frac{1}{N^{k/2+1/2} \sqrt{\eta \eta_a}},$$

whenever $n \notin \{i_1, \dots, i_k\}$. Thus for most of the $\mathcal{O}(N^{k+1})$ terms in the sum in eq. (55) we have the improved bound, while for $\mathcal{O}(N^k)$ terms, where $n = i_l$ for some l , we use the weaker bound and we find that

$$\begin{aligned} N^{-k/2} \sum_{i_1, \dots, i_k}^{\sim} \sum_n \frac{1}{N} \mathbf{E} \frac{\partial [G_{n i_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)}]}{\partial h_{i_1 n}} \\ = \frac{1}{N^{k/2+1} z_1} \sum_{n, i_1, \dots, i_k}^{\sim} \left[-m(z_1) \mathbf{E} G_{i_1 i_2}^{(1)} \dots G_{i_k i_1}^{(k)} - m(z_k) \mathbf{E} G_{n i_2}^{(1)} \dots G_{i_k n}^{(k)} \right] + \frac{1}{z_1} \mathcal{O}_{\prec} \left(\sum_{a=1}^k \frac{1}{\sqrt{N \eta \eta_a}} \right). \end{aligned} \quad (59)$$

It remains to estimate the error R . To do so we have to compute the second derivatives

$$\frac{\partial^2 \left[G_{ni_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} \right]}{\partial h_{i_1 n}^2}$$

which is a polynomial in $G_{ab}^{(l)}$ for $l \in \{1, \dots, k\}$, $a, b \in \{i_1, \dots, i_k, n\}$ of total degree $k+2$ with at most 2 diagonal factors for $n \notin \{i_1, \dots, i_k\}$, and otherwise with at most 3 diagonal factors in every monomial. These factors each satisfy the entry-wise local law (22), but now we need these estimates even uniformly for all $|h_{i_1 n}| \leq N^{\tau-1/2}$ which does not directly follow from the concept of stochastic domination. To circumvent this technical issue, we need to explicitly display the dependence of the resolvents $G^{(l)}$ on $h_{i_1 n}$. We therefore write \tilde{H} for the matrix \hat{H} with the (i_1, n) and (n, i_1) entries set to 0 and $\tilde{G}^{(l)} = (\tilde{H} - z_l)^{-1}$. Note that $\tilde{G}^{(l)}$ is independent of $h_{i_1 n}$. Since $\tilde{G}^{(l)}$ is the resolvent of a generalized Wigner matrix, from [15, 14] we have the usual resolvent estimates (21)–(22) for $\tilde{G}^{(l)}$. Moreover, if $i_1 \neq n$, then by the resolvent identity

$$G_{ab}^{(l)} = \tilde{G}_{ab}^{(l)} - h_{i_1 n} \left[\tilde{G}_{an}^{(l)} \tilde{G}_{i_1 b}^{(l)} + \tilde{G}_{a i_1}^{(l)} \tilde{G}_{nb}^{(l)} \right] + h_{i_1 n}^2 \left[\tilde{G}_{an}^{(l)} \tilde{G}_{i_1 n}^{(l)} G_{i_1 b}^{(l)} + \tilde{G}_{an}^{(l)} \tilde{G}_{i_1 i_1}^{(l)} G_{nb}^{(l)} + \tilde{G}_{a i_1}^{(l)} \tilde{G}_{nn}^{(l)} G_{i_1 b}^{(l)} + \tilde{G}_{a i_1}^{(l)} \tilde{G}_{n i_1}^{(l)} G_{nb}^{(l)} \right]$$

and we can estimate

$$\max_{a \neq b} \sup_{|h_{i_1 n}| \leq N^{-1/2+\tau}} G_{ab}^{(l)} \prec \frac{N^\tau}{\sqrt{N\eta_l}}, \quad \max_a \sup_{|h_{i_1 n}| \leq N^{-1/2+\tau}} G_{aa}^{(l)} \prec 1$$

whenever $\tau < 1/12$ where we used the trivial bound $G_{ab}^{(l)} \leq 1/\eta_l \leq N^{2/3}$. On the other hand, if $i_1 = n$, then we have

$$G_{ab}^{(l)} = \tilde{G}_{ab}^{(l)} - h_{nn} \tilde{G}_{an}^{(l)} \tilde{G}_{nb}^{(l)} + h_{nn}^2 \tilde{G}_{an}^{(l)} \tilde{G}_{nn}^{(l)} G_{nb}^{(l)}$$

and therefore again

$$\max_{a \neq b} \sup_{|h_{nn}| \leq N^{-1/2+\tau}} G_{ab}^{(l)} \prec \frac{N^\tau}{\sqrt{N\eta_l}}, \quad \max_a \sup_{|h_{nn}| \leq N^{-1/2+\tau}} G_{aa}^{(l)} \prec 1$$

whenever $\tau < 1/12$. Therefore

$$\sup_{|h_{i_1 n}| < N^{-1/2+\tau}} \left| \frac{\partial^2 \left[G_{ni_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} \right]}{\partial h_{i_1 n}^2} \right| \prec \sum_{a=1}^k \frac{N^{k\tau} N^{-k/2}}{\sqrt{\eta_a}}$$

and we can conclude

$$\frac{1}{N^{k/2} z_1} \sum_{i_1, \dots, i_k} \sum_n \mathbf{E} |h_{i_1 n}|^3 \sup_{|h_{i_1 n}| < N^{-1/2+\tau}} \left| \frac{\partial^2 \left[G_{ni_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} \right]}{\partial h_{i_1 n}^2} \right| \prec \sum_{a=1}^k \frac{N^{k\tau} N^{-1/2}}{\sqrt{\eta_a}}. \quad (60)$$

We can now pick $\tau = \min\{\frac{1}{12}, \frac{\epsilon}{k}\}$ to have a final estimate of order

$$\sum_{a=1}^k \frac{N^\epsilon}{\sqrt{N\eta_a}}$$

for the error originating from the last term in the truncated cumulant expansion (56). The remaining error

$$\mathbf{E} \left| h_{i_1 n}^3 \mathbf{1}(|h_{i_1 n}| > N^{\tau-1/2}) \sup_{|h_{i_1 n}|} \left| \frac{\partial^2 \left[G_{ni_2}^{(1)} G_{i_2 i_3}^{(2)} \dots G_{i_k i_1}^{(k)} \right]}{\partial h_{i_1 n}^2} \right| \right| \quad (61)$$

is negligible for any fixed k since the expectation is smaller than any power of $N^{-\tau}$ due to the arbitrary polynomial decay (5).

Putting together (59), the identity

$$z_1 + m(z_1) + m(z_k) = \frac{m(z_1)m(z_k) - 1}{m(z_1)}$$

and the estimates on R from (60)–(61) we have shown that

$$N^{-k/2} \sum_{i_1 \neq \dots \neq i_k} \mathbf{E} G_{i_1 i_2}^{(1)} \dots G_{i_k i_1}^{(k)} = \frac{m(z_1)}{1 - m(z_1)m(z_k)} \mathcal{O} \left(\sum_{a=1}^k \frac{N^\epsilon}{\sqrt{N\eta_a}} \right) = \mathcal{O} \prec \left(\sum_{a=1}^k \frac{N^\epsilon}{(\eta_1 + \eta_k) \sqrt{N\eta_a}} \right).$$

Since the lhs. of this estimate is cyclic in i_1, \dots, i_k , we can replace $\eta_1 + \eta_k$ in the error term by $\max_a \eta_a$.

For the proof of eq. (54) we follow essentially the same steps but for the last $a = k - 1$ term we find

$$-(G_{i_{k-1}i_1}^{(k-1)} G_{ni_k}^{(k-1)} + G_{i_{k-1}n}^{(k-1)} G_{i_1i_k}^{(k-1)}) G_{ni_2}^{(1)} \dots G_{i_{k-2}i_{k-1}}^{(k-2)} \prec \frac{1}{N^{k/2} \sqrt{\eta \eta_{k-1}}}$$

instead of eq. (58). Consequently, eq. (59) becomes

$$\begin{aligned} & N^{-(k+1)/2} \sum_{i_1 \neq \dots \neq i_k} \mathbf{E} G_{i_1i_2}^{(1)} \dots G_{i_{k-1}i_k}^{(k-1)} \\ &= \frac{1}{N^{(k+1)/2+1} z_1} \sum_{n \neq i_1 \neq \dots \neq i_k} \left[-m(z_1) \mathbf{E} G_{i_1i_2}^{(1)} \dots G_{i_{k-1}i_k}^{(k-1)} \right] + \frac{1}{z_1} \mathcal{O}_{\prec} \left(\sum_{a=1}^{k-1} \frac{N^\epsilon}{\sqrt{N \eta \eta_a}} \right) \end{aligned}$$

from which eq. (54) follows immediately.

For the last claim, note that none of the estimates above relied on the order of the indices of any $G^{(l)}$ and the same bound holds true in the case of any transpositions.

The proof of the Hermitian case is similar, but the cumulant expansion has to be replaced by a complex variant (as in, e.g. [18, Lemma 7.1]). \square

Next, we note that the bounds (53)–(54) also hold true without taking expectations:

Corollary 12. *In the setup of Lemma 11, for closed cycles of length $k \geq 2$ we have that*

$$N^{-k/2} \sum_{i_1, \dots, i_k}^{\sim} G_{i_1i_2}^{(1)} \dots G_{i_{k-1}i_k}^{(k-1)} G_{i_ki_1}^{(k)} \prec \frac{1}{(\max_a \eta_a) \sqrt{N \eta_1 \dots \eta_k}} \sum_{a=1}^k \frac{1}{\sqrt{\eta_a}}, \quad (62)$$

and for open cycles of any length $k > 1$ we have that

$$N^{-(k+1)/2} \sum_{i_1, \dots, i_k}^{\sim} G_{i_1i_2}^{(1)} \dots G_{i_{k-1}i_k}^{(k-1)} \prec \frac{1}{\sqrt{N \eta_1 \dots \eta_{k-1}}} \sum_{a=1}^{k-1} \frac{1}{\sqrt{\eta_a}}. \quad (63)$$

Proof. We note that the fluctuation averaging analysis from [13, Proof of Prop. 5.3 in Sections 6–7] does not rely on the fact $z_1 = \dots = z_k$ and therefore also applies to the present case. \square

The following lemma shows an asymptotic Wick theorem for X 's, i.e. that higher moments of X to leading order only involve pairings:

Lemma 13. *For $k \geq 2$ and $z_1, \dots, z_k \in \mathbb{C}$ with $|\Im z_l| = \eta_l > 0$ we have that*

$$\mathbf{E}[X(z_1) \dots X(z_k)] = \sum_{\pi \in P_2([k])} \prod_{\{a,b\} \in \pi} \mathbf{E}[X(z_a) X(z_b)] + \mathcal{O}_{\prec} \left(\frac{1}{\sqrt{N \eta_1 \dots \eta_k}} \sum_{a \neq b} \frac{1}{(\eta_a + \eta_b) \sqrt{\eta_a}} \right), \quad (64)$$

where $[k] := \{1, \dots, k\}$ and $P_2(L)$ are the partitions of a set L into subsets of size 2.

Proof. For definiteness we prove the real symmetric case. Since the argument relies on counting pairings, the proof of the complex Hermitian case is very similar and we omit it. We have to compute

$$\mathbf{E}_1 \prod_{l=1}^k \left[\sum_{i_l \neq j_l} h_{i_l} G_{i_l j_l}^{(l)} h_{j_l} + \sum_{i_l} \left(h_{i_l}^2 - \frac{1}{N} \right) G_{i_l i_l}^{(l)} \right] = \sum_{L \subset [k]} \mathbf{E}_1 \left[\left(\prod_{l \in L} \sum_{i_l \neq j_l} h_{i_l} G_{i_l j_l}^{(l)} h_{j_l} \right) \left(\prod_{l \notin L} \sum_{i_l} \left(h_{i_l}^2 - \frac{1}{N} \right) G_{i_l i_l}^{(l)} \right) \right],$$

where $[k] = \{1, \dots, k\}$ and $\mathbf{E}_1 = \mathbf{E}(\cdot | H^{(1)}) = \mathbf{E}(\cdot | \widehat{H})$ and we recall that $G^{(l)}$ is independent of h . We already know from eq. (35) and Lemma 9 that the leading order of this expression is at most $N^{-k/2}$. In order to have non-zero expectation we have to pair any h_{i_l} and h_{j_l} with at least some other h_{i_m} or h_{j_m} . An easy counting argument using the bound $G_{i_l j_l}^{(l)} \prec (N \eta_l)^{-1/2}$ shows that for any $L \subset [k]$ the corresponding L -term is at most of order

$$N^{-(k+1)/2} \prod_{l \in L} \eta_l^{-1/2}$$

whenever any three or more h_i 's are paired. This already shows that we can restrict our attention to pairings and in particular odd moments asymptotically are of lower order.

Starting from some h_{i_l} with $l \notin L$ we have to pair it either to another h_{i_m} with $m \notin L$, or some h_{i_m} or h_{j_m} with $m \in L$. In the former case we have a closed pairing with expectation

$$\mathbf{E}_1 \left[(h_{i_l}^2 - 1/N)(h_{i_l}^2 - 1/N) G_{i_l i_l}^{(l)} G_{i_l i_l}^{(m)} \right] = \frac{\sigma_4 - 1}{N^2} G_{i_l i_l}^{(l)} G_{i_l i_l}^{(m)}.$$

In the latter case, say we paired h_{i_l} to h_{i_m} , we have to continue the pairing process by pairing h_{j_m} with another h_{i_k} or h_{j_k} with $k \in L$ etc., until we reach another h_{i_n} with $n \notin L$. This expression represents an open cycle as in (63) and is therefore subleading.

On the other hand, starting from some h_{i_l} or h_{j_l} with $l \in L$, and continue the pairings as in the previous paragraph until we pair to an h_{i_m} with $m \notin L$ which results in an open cycle as in (63) and is subleading. Therefore we only have to consider closed cycles of the pure L -type, from which, due to (62), only those of length 2 are leading. That means that pairing h_{i_l} to h_{i_m} automatically forces a pairing of h_{j_l} and h_{j_m} , and that a pairing of h_{i_l} to h_{j_m} automatically forces a pairing of h_{j_l} and h_{i_m} . These give the leading contribution of

$$\mathbf{E}_1 \left[h_{i_l} G_{i_l j_l}^{(l)} h_{j_l} h_{j_l} G_{j_l i_l}^{(m)} h_{i_l} + h_{i_l} G_{i_l j_l}^{(l)} h_{j_l} h_{i_l} G_{i_l j_l}^{(m)} h_{j_l} \right] = \frac{G_{i_l j_l}^{(l)} G_{i_l j_l}^{(m)}}{N^2}.$$

The above findings allow us to conclude that

$$\begin{aligned} N^{k/2} \mathbf{E}_1 & \left[\left(\prod_{l \in L} \sum_{i_l \neq j_l} h_{i_l} G_{i_l j_l}^{(l)} h_{j_l} \right) \left(\prod_{l \notin L} \sum_{i_l} \left(h_{i_l}^2 - \frac{1}{N} \right) G_{i_l i_l}^{(l)} \right) \right] \\ &= N^{k/2} \mathbf{E}_1 \left(\prod_{l \in L} \sum_{i_l \neq j_l} h_{i_l} G_{i_l j_l}^{(l)} h_{j_l} \right) \mathbf{E}_1 \left(\prod_{l \notin L} \sum_{i_l} \left(h_{i_l}^2 - \frac{1}{N} \right) G_{i_l i_l}^{(l)} \right) + \mathcal{O}_{\prec}(\Psi) \\ &= N^{k/2} \left(\sum_{\pi \in P_2(L)} \prod_{\{a,b\} \in \pi} \sum_{i \neq j} \frac{G_{ij}^{(a)} G_{ij}^{(b)} + G_{ij}^{(a)} G_{ji}^{(b)}}{N^2} \right) \left(\sum_{\pi \in P_2([k] \setminus L)} \prod_{\{a,b\} \in \pi} \frac{\sigma_4 - 1}{N^2} \sum_i G_{ii}^{(a)} G_{ii}^{(b)} \right) + \mathcal{O}_{\prec}(\Psi) \\ &= N^{k/2} \left(\sum_{\pi \in P_2(L)} \prod_{\{a,b\} \in \pi} \frac{2}{N} \frac{m(z_a)^2 m(z_b)^2}{1 - m(z_a) m(z_b)} \right) \left(\sum_{\pi \in P_2([k] \setminus L)} \prod_{\{a,b\} \in \pi} \frac{\sigma_4 - 1}{N} m(z_a) m(z_b) \right) + \mathcal{O}_{\prec}(\Psi), \end{aligned}$$

where in the last step we used Lemma 9 and we introduced the error term

$$\Psi = \frac{1}{\sqrt{N \eta_1 \dots \eta_k}} \sum_{a \neq b} \frac{1}{(\eta_a + \eta_b) \sqrt{\eta_a}}.$$

We now recognize the last expression as the sum over products of pairs of $\mathbf{E}[X(z_a)X(z_b)]$, completing the proof. \square

We now have all ingredients to compute

$$\begin{aligned} & \left(-\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) [X(z) - \sqrt{N} h_{11}] d\eta dx \right)^k \\ &= \sum_{j=0}^k \binom{k}{j} (\sqrt{N} h_{11})^{k-j} \left(\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) d\eta dx \right)^{k-j} \left(-\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) X(z) d\eta dx \right)^j. \end{aligned}$$

Recall that h_{11} and X are independent. From Lemma 13 we can conclude that

$$\begin{aligned} \mathbf{E} \left(-\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) X(z) d\eta dx \right)^j &= \sum_{\pi \in P_2([j])} (2V_{f,1} + (\sigma_4 - 1)V_{f,2})^{j/2} + \mathcal{O}_{\prec}(N^{-1/6}) \\ &= (j-1)!! (2V_{f,1} + (\sigma_4 - 1)V_{f,2})^{j/2} + \mathcal{O}_{\prec}(N^{-1/6}) \end{aligned}$$

for even j and

$$\mathbf{E} \left(-\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) X(z) d\eta dx \right)^j = \mathcal{O}_{\prec}(N^{-1/6})$$

for odd j . If h_{11} follows a normal distribution, then $\mathbf{E} h_{11}^{k-j} = (k-j-1)!! (s_{11}/N)^{(k-j)/2}$ whenever $k-j$ is even and $\mathbf{E} h_{11}^{k-j} = 0$, otherwise. Therefore, since

$$(j-1)!!(k-j-1)!! \binom{k}{j} = (k-1)!! \binom{k/2}{j/2}$$

for even j, k , we have that

$$\mathbf{E} \left(-\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) [X(z) - \sqrt{N} h_{11}] d\eta dx \right)^k = (k-1)!! [2V_{f,1} + (\sigma_4 - 1)V_{f,2} + s_{11}V_{f,3}]^{k/2} + \mathcal{O}_{\prec} (N^{-1/6}) \quad (65)$$

whenever k is even and

$$\mathbf{E} \left(-\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) [X(z) - \sqrt{N} h_{11}] d\eta dx \right)^k = \mathcal{O}_{\prec} (N^{-1/6})$$

otherwise.

For the case of complex Hermitian H we can follow the same argument and ultimately find that eq. (65) becomes

$$\mathbf{E} \left(-\Im \int_{\mathbb{R}} \int_{\eta_0}^{10} g(z) [X(z) - \sqrt{N} h_{11}] d\eta dx \right)^k = (k-1)!! [V_{f,1} + |\sigma_2|^2 V_{\sigma_2} + (\sigma_4 - 1)V_{f,2} + s_{11}V_{f,3}]^{k/2} + \mathcal{O}_{\prec} (N^{-1/6}).$$

Finally, we remark that the same proof also works in the case of \tilde{f}_N and we basically only have to replace $V_{f,3}$ by $\tilde{V}_{f,3}$.

APPENDIX A. COMPARISON TO GAUSSIAN FREE FIELD

In this section we investigate to what extent our main result on the Gaussian fluctuation of linear statistics of H and its minor \hat{H} is consistent with the Gaussian free field (GFF) limit proved in [7, 8, 30] for real symmetric matrices. In these papers the joint fluctuations of the spectral counting functions of minors were shown to converge to a GFF in the large N limit, assuming that the sizes of the minors asymptotically differed by cN . Our result corresponds to the difference of the linear statistics of two minors whose sizes differ only by one. The fluctuation is only of order $N^{-1/2}$ and it is not visible on the macroscopic scale studied in [7, 8, 30]. Nevertheless, one may *formally* apply these macroscopic result to our case. Here we show that this naive extension indeed provides the correct order of magnitude and also the correct variance of the fluctuations, but does not identify their precise distribution.

For comparability with [7, 8, 30] assume a constant variance on the diagonal and constant fourth moment on the off-diagonal, i.e., $\mathbf{E} h_{ii}^2 = \mathbf{E} h_{11}^2 = s_{11}/N$ and $\mathbf{E} h_{ij}^4 = \sigma_4/N^2$ for all $i \neq j$. First we recall the main result of [30] which is based on [7], where the corresponding formula was first proved for monomial test functions. Given an $N \times N$ Wigner matrix H , we denote the consecutive lower right minors by $H_n := (H_{jk})_{j,k=N-n+1}^N$. A special case of Theorem 2.2 of [30] then asserts that for any $f \in H^{5.5+\epsilon}(\mathbb{R})$ and for any $x, y \in (0, 1]$, the covariance of linear statistics of two nested minors of size Nx and Ny is asymptotically given by

$$\begin{aligned} C_f(x, y) &:= \lim_{N \rightarrow \infty} \mathbf{Cov} (\mathrm{Tr} f(H_{[xN]}), \mathrm{Tr} f(H_{[yN]})) \\ &= \frac{1}{\pi^2} \oint_{|z|^2=x} \oint_{\substack{|w|^2=y \\ \Im z > 0}} f' \left(z + \frac{x}{z} \right) f' \left(w + \frac{y}{w} \right) \log \left| \frac{x \wedge y - zw}{x \wedge y - z\bar{w}} \right| \left(1 - \frac{x}{z^2} \right) \left(1 - \frac{y}{w^2} \right) dw dz \\ &\quad + \frac{s_{11} - 2}{x \vee y} \left(\frac{1}{2\pi} \int_{-2\sqrt{x}}^{2\sqrt{x}} \frac{sf(s)}{\sqrt{4x - s^2}} ds \right) \left(\frac{1}{2\pi} \int_{-2\sqrt{y}}^{2\sqrt{y}} \frac{tf(t)}{\sqrt{4y - t^2}} dt \right) \\ &\quad + \frac{\sigma_4 - 3}{2(x \vee y)^2} \left(\int_{-2\sqrt{x}}^{2\sqrt{x}} \frac{2x - s^2}{\pi \sqrt{4x - s^2}} f(s) ds \right) \left(\int_{-2\sqrt{y}}^{2\sqrt{y}} \frac{2y - t^2}{\pi \sqrt{4y - t^2}} f(t) dt \right) \end{aligned} \quad (66)$$

where the z and w integrations in the first term are on the semicircular arcs in counterclockwise order.

Recalling our previous notation $H = H_N$ and $\widehat{H} = H_{N-1}$, in our Theorem 1 we derived a formula for the rescaled variance

$$\begin{aligned} D_{N,f} &:= N \mathbf{Var}[\mathrm{Tr} f(H_N) - \mathrm{Tr} f(H_{N-1})] \\ &= N [\mathbf{Cov}(\mathrm{Tr} f(H_N), \mathrm{Tr} f(H_N)) - \mathbf{Cov}(\mathrm{Tr} f(H_N), \mathrm{Tr} f(H_{N-1})) \\ &\quad - \mathbf{Cov}(\mathrm{Tr} f(H_{N-1}), \mathrm{Tr} f(H_N)) + \mathbf{Cov}(\mathrm{Tr} f(H_{N-1}), \mathrm{Tr} f(H_{N-1}))], \end{aligned}$$

which corresponds to

$$N \left[C_f(1, 1) - C_f\left(1, 1 - \frac{1}{N}\right) - C_f\left(1 - \frac{1}{N}, 1 - \frac{1}{N}\right) + C_f\left(1 - \frac{1}{N}, 1 - \frac{1}{N}\right) \right],$$

suggesting that we should compare our result to the limit

$$D_f := \lim_{\epsilon \rightarrow 0} \frac{C_f(1, 1) - C_f(1, 1 - \epsilon) - C_f(1 - \epsilon, 1) - C_f(1 - \epsilon, 1 - \epsilon)}{\epsilon}. \quad (67)$$

Note that this latter formula is the renormalized derivative of the Gaussian free field $\phi_x(f)$ with covariance $C_f(x, y)$ at $x = 1$:

$$D_f = \lim_{\epsilon \rightarrow 0} \mathbf{Var} \frac{\phi_1(f) - \phi_{1-\epsilon}(f)}{\sqrt{\epsilon}}.$$

In the following theorem we compare the field

$$\psi_x^{(N)}(f) := \mathrm{Tr} f(H_{[xN]}) - \mathbf{E} \mathrm{Tr} f(H_{[xN]})$$

defined by our linear eigenvalue statistics to the Gaussian free field $\phi_x(f)$.

Theorem 14. *Let H be real symmetric Wigner matrices satisfying the conditions from Theorem 1 and additionally assume that $\mathbf{E} h_{ii}^2 = \mathbf{E} h_{11}^2 = s_{11}/N$ and $\mathbf{E} h_{ij}^4 = \sigma_4/N^2$ for all $i \neq j$. Then for any $f \in H^2(\mathbb{R})$ the centered random variables*

$$X_f := \lim_{\epsilon \rightarrow 0} \frac{\phi_1(f) - \phi_{1-\epsilon}(f)}{\sqrt{\epsilon}} \quad \text{and} \quad Y_f := \lim_{N \rightarrow \infty} \frac{\psi_1^{(N)}(f) - \psi_{1-1/N}^{(N)}(f)}{\sqrt{1/N}}$$

are well defined (the limit is in distribution sense) and they have the same variance

$$\mathbf{E} X_f^2 = \mathbf{E} Y_f^2 = 2 \int_{-2}^2 f'(s)^2 \rho(s) ds + (\sigma_4 - 3) \left(\int_{-2}^2 s f'(s) \rho(s) ds \right)^2 + (s_{11} - 2) \left(\int_{-2}^2 f'(s) \rho(s) ds \right)^2. \quad (68)$$

Moreover, the distributions of X_f and Y_f agree if and only if h_{11} follows a Gaussian distribution.

Proof. The variance formula for Y_f follows immediately from Theorem 1. Moreover, we know from Theorem 2 that Y_f follows a Gaussian distribution if and only if h_{11} follows a Gaussian distribution.

In order to prove that X_f is well defined and follows a Gaussian distribution, it suffices to check that D_f is finite. To do so, we treat the three terms of $C_f(x, y)$ from (66) separately, which for convenience we call $C_f(x, y) = C_f^{(1)}(x, y) + C_f^{(2)}(x, y) + C_f^{(3)}(x, y)$. It is easy to check that

$$\lim_{\epsilon \rightarrow 0} \frac{C_f^{(2)}(1, 1) - C_f^{(2)}(1, 1 - \epsilon) - C_f^{(2)}(1 - \epsilon, 1) - C_f^{(2)}(1 - \epsilon, 1 - \epsilon)}{\epsilon} = (s_{11} - 2) \left(\int_{-2}^2 f'(s) \rho(s) ds \right)^2$$

and that

$$\lim_{\epsilon \rightarrow 0} \frac{C_f^{(3)}(1, 1) - C_f^{(3)}(1, 1 - \epsilon) - C_f^{(3)}(1 - \epsilon, 1) - C_f^{(3)}(1 - \epsilon, 1 - \epsilon)}{\epsilon} = (\sigma_4 - 3) \left(\int_{-2}^2 s f'(s) \rho(s) ds \right)^2.$$

For the computation of $C_f^{(1)}(x, y)$ we now substitute $z = \sqrt{x}e^{i\phi}$ and $w = \sqrt{y}e^{i\psi}$ with $\phi, \psi \in [0, \pi]$, so that

$$C_f^{(1)}(x, y) = \frac{4\sqrt{xy}}{\pi^2} \int_0^\pi \int_0^\pi f'(2\sqrt{x} \cos \phi) f'(2\sqrt{y} \cos \psi) \log \left| \frac{x \wedge y - \sqrt{xy}e^{i(\phi+\psi)}}{x \wedge y - \sqrt{xy}e^{i(\phi-\psi)}} \right| \sin \phi \sin \psi d\psi d\phi$$

and after a further substitution of $2\sqrt{x} \cos \phi = s$ and $2\sqrt{y} \cos \psi = t$ and simple algebraic manipulation we arrive at

$$C_f^{(1)}(x, y) = \frac{1}{\pi^2} \int_{-2\sqrt{x}}^{2\sqrt{x}} \int_{-2\sqrt{y}}^{2\sqrt{y}} f'(s) f'(t) \operatorname{arctanh} \frac{\sqrt{(4x-s^2)(4y-t^2)}}{2(x+y)-st} dt ds.$$

To keep the notation relatively short we now introduce

$$a_{x,y}(s,t) := \operatorname{arctanh} \frac{\sqrt{(4x-s^2)(4y-t^2)}}{2(x+y)-st} = \operatorname{arctanh} \sqrt{1 - \frac{(x-y)^2 + (t-s)(xt-ys)}{(x+y)^2 - (x+y)st + s^2t^2/4}}$$

and we claim that

$$\frac{a_{1,1}(s,t) - a_{1,1-\epsilon}(s,t) - a_{1-\epsilon,1}(s,t) + a_{1-\epsilon,1-\epsilon}(s,t)}{\epsilon} \approx \pi \delta(s-t) \sqrt{4-t^2}$$

for any fixed $s, t \in [-2, 2]$ in the $\epsilon \rightarrow 0$ limit. Firstly, one readily checks that when $|s-t| \gg \epsilon$, then

$$\lim_{\epsilon \rightarrow 0} \frac{a_{1,1}(s,t) - a_{1,1-\epsilon}(s,t) - a_{1-\epsilon,1}(s,t) + a_{1-\epsilon,1-\epsilon}(s,t)}{\epsilon} = 0.$$

Secondly, when $|x-y| \leq \epsilon$ and $|s-t| \leq M\epsilon$ for some large but fixed M , then a series expansion gives

$$a_{x,y}(s,t) = \log 2 - \frac{1}{2} \log \frac{(x-y)^2 + (t-s)(xt-ys)}{(x+y)^2 - (x+y)st + s^2t^2/4} - \frac{1}{4} \frac{(x-y)^2 + (t-s)(xt-ys)}{(x+y)^2 - (x+y)st + s^2t^2/4} + \mathcal{O}(\epsilon^2),$$

assuming, additionally, that $|s| \leq 2\sqrt{x}(1-\delta)$, $|t| \leq 2\sqrt{y}(1-\delta)$ with some fixed $\delta > 0$. It can now be checked via an explicit integration that

$$\int_{|s-t| < M\epsilon} \frac{a_{1,1}(s,t) - a_{1,1-\epsilon}(s,t) - a_{1-\epsilon,1}(s,t) + a_{1-\epsilon,1-\epsilon}(s,t)}{\epsilon} ds = \pi \sqrt{4-t^2} + \mathcal{O}(\epsilon)$$

for fixed t , proving the claim. We can conclude that

$$\begin{aligned} \frac{C_f^{(1)}(1,1) - C_f^{(1)}(1,1-\epsilon) - C_f^{(1)}(1-\epsilon,1) - C_f^{(1)}(1-\epsilon,1-\epsilon)}{\epsilon} &= \frac{1}{\pi^2} \int_{-2}^2 \int_{-2}^2 f'(s)f'(t)\delta(s-t)\pi\sqrt{4-t^2} ds dt + \mathcal{O}(\epsilon) \\ &= 2 \int_{-2}^2 f'(t)^2 \rho(t) dt + \mathcal{O}(\epsilon), \end{aligned}$$

where we used that $f' \in L^2$ and therefore the integral over the neglected area where $|s| > 2\sqrt{x}(1-\delta)$ or $|t| > 2\sqrt{y}(1-\delta)$ does not contribute to leading order. Thus

$$D_f = 2 \int_{-2}^2 f'(s)^2 \rho(s) ds + (\sigma_4 - 3) \left(\int_{-2}^2 s f'(s) \rho(s) ds \right)^2 + (s_{11} - 2) \left(\int_{-2}^2 f'(s) \rho(s) ds \right)^2,$$

completing the proof of (68). In particular, the limit defining X_f exists and is Gaussian. Finally, the existence of the limit defining Y_f follows from the moment calculations in the proof of Theorem 2 and assumption (5) on the moments of h_{11} that together also guarantee tightness. This completes the proof of the theorem. \square

REFERENCES

- [1] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An Introduction to Random Matrices*. Cambridge studies in advanced mathematics. Cambridge University Press, Leiden, 2009.
- [2] W. G. Anderson and O. Zeitouni. A CLT for a band matrix model. *Probab. Theory Related Fields*, 134(2):283–338, 2006.
- [3] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999.
- [4] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the second row of a Young diagram under Plancherel measure. *Geom. Funct. Anal.*, 10(4):702–731, 2000.
- [5] Z. Bao, G. Pan, and W. Zhou. Central limit theorem for partial linear eigenvalue statistics of Wigner matrices. *J. Stat. Phys.*, 150:88–129, Jan. 2013.
- [6] Y. Baryshnikov. GUEs and queues. *Probab. Theory Related Fields*, 119(2):256–274, 2001.
- [7] A. Borodin. CLT for spectra of submatrices of Wigner random matrices. *Mosc. Math. J.*, 14(1):29–38, 170, 2014.
- [8] A. Borodin. CLT for spectra of submatrices of Wigner random matrices, II: Stochastic evolution. In *Random matrix theory, interacting particle systems, and integrable systems*, volume 65 of *Math. Sci. Res. Inst. Publ.*, pages 57–69. Cambridge Univ. Press, New York, 2014.
- [9] A. Borodin and V. Gorin. Lectures on integrable probability. *ArXiv e-print 1212.3351*, Dec. 2012.
- [10] A. Borodin and V. Gorin. General β -Jacobi corners process and the Gaussian free field. *Comm. Pure Appl. Math.*, 68(10):1774–1844, 2015.
- [11] A. Borodin, A. Okounkov, and G. Olshanski. Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.*, 13(3):481–515, 2000.
- [12] A. Bufetov. Kerov’s interlacing sequences and random matrices. *J. Math. Phys.*, 54(11):113302, 2013.
- [13] L. Erdős, A. Knowles, and H.-T. Yau. Averaging fluctuations in resolvents of random band matrices. *Ann. Henri Poincaré*, 14:1837–1926, Dec. 2013.
- [14] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. The local semicircle law for a general class of random matrices. *Electron. J. Probab.*, 18(59):1–58, 2013.

- [15] L. Erdős, H.-T. Yau, and J. Yin. Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.*, 229(3):1435 – 1515, 2012.
- [16] W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1997.
- [17] V. Gorin and L. Zhang. Work in preperation.
- [18] Y. He and A. Knowles. Mesoscopic eigenvalue statistics of Wigner matrices. *ArXiv e-print 1603.01499*, Mar. 2016.
- [19] B. Helffer and J. Sjöstrand. *Equation de Schrödinger avec champ magnétique et équation de Harper*, pages 118–197. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.
- [20] V. Ivanov and G. Olshanski. Kerov’s central limit theorem for the Plancherel measure on Young diagrams. In *Symmetric functions 2001: surveys of developments and perspectives*, pages 93–151. Springer, 2002.
- [21] I. Jana, K. Saha, and A. B. Soshnikov. Fluctuations of linear eigenvalue statistics of random band matrices. *Teor. Veroyatnost. i Primenen.*, 60(3):553–592, 2015.
- [22] J. K. Johansson. Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. *Comm. Math. Phys.*, 215(3):683–705, 2001.
- [23] K. Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math.*, 153(1):259–296, 2001.
- [24] K. Johansson and E. Nordenstam. Eigenvalues of GUE minors. *Electron. J. Probab.*, 11(50):1342–1371, 2006.
- [25] T. Johnson and S. Pal. Cycles and eigenvalues of sequentially growing random regular graphs. *Ann. Probab.*, 42(4):1396–1437, 07 2014.
- [26] V. Kargin. Limit theorems for linear eigenvalue statistics of overlapping matrices. *Electron. J. Probab.*, 20:30 pp., 2015.
- [27] S. Kerov and A. Vershik. Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. *Sov. Math. Dokl.*, 18:527–531, 1977.
- [28] S. V. Kerov. The asymptotics of root separation for orthogonal polynomials. *St. Petersburg Math. J.*, 5:925–941, 1993.
- [29] A. M. Khorunzhy, B. A. Khoruzhenko, and L. A. Pastur. Asymptotic properties of large random matrices with independent entries. *J. Math. Phys.*, 37(10), 1996.
- [30] L. Li, M. Reed, and A. Soshnikov. Central limit theorem for linear eigenvalue statistics for submatrices of Wigner random matrices. *ArXiv e-print 1504.05933*, Apr. 2015.
- [31] B. Logan and L. Shepp. A variational problem for random Young tableaux. *Adv. in Math.*, 26(2):206 – 222, 1977.
- [32] A. Lytova and L. Pastur. Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. *Ann. Probab.*, 37(5):1778–1840, 09 2009.
- [33] A. Okounkov. Random matrices and random permutations. *Int. Math. Res. Not.*, 2000(20):1043–1095, 2000.
- [34] G. I. Olshanski. *Kirillov’s seminar on representation theory*, volume 181. American Mathematical Soc., 1998.
- [35] L. Pastur. Limiting laws of linear eigenvalue statistics for Hermitian matrix models. *J. Math. Phys.*, 47(10), 2006.
- [36] M. Prähofer and H. Spohn. Universal distributions for growth processes in $1 + 1$ dimensions and random matrices. *Phys. Rev. Lett.*, 84:4882–4885, May 2000.
- [37] M. Shcherbina. Central limit theorem for linear eigenvalue statistics of the Wigner and sample covariance random matrices. *Zh. Mat. Fiz. Anal. Geom.*, 7(2):176–192, 197, 199, 2011.
- [38] M. Shcherbina. Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime. *J. Stat. Phys.*, 151(6):1004–1034, 2013.
- [39] P. Sosoe and P. Wong. Regularity conditions in the CLT for linear eigenvalue statistics of Wigner matrices. *Adv. in Math.*, 249:37–87, 2013.
- [40] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159(1):151–174, 1994.